Section 2.4 Supplementary problems.

Problem 1 Suppose

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

has the property that, for any $\vec{X} \in \mathbb{R}^2$,

$$T(\vec{x}) \;=\; A\vec{x}$$

for some 2×2 matrix A.

Prove that

$$T(\vec{x} + \vec{y}) \;=\; T(\vec{x}) + T(\vec{y}) \quad \text{and} \quad T(k\vec{x}) = kT(\vec{x})$$

where $\vec{x}$ and $\vec{y}$ are arbitrary elements of $\mathbb{R}^2$ and $k \in \mathbb{R}$ is an arbitrary scalar.

Problem 2 Now assume that, for arbitrary vectors $\vec{x}, \vec{y} \in \mathbb{R}^2$ and arbitrary scalar $k \in \mathbb{R}$,

$$T(\vec{x} + \vec{y}) \;=\; T(\vec{x}) + T(\vec{y}) \quad \text{and} \quad T(k\vec{x}) = kT(\vec{x})$$

Prove that there is a 2×2 matrix A with the property that

$$T(\vec{x}) = A\vec{x}$$

for any $x \in \mathbb{R}^2$