Section 2.4 Hints

Problem 28 Try matrices with a single element equal to 1 and all others zero. One of them will work.

The implication here is that, unlike multiplication with numbers, the product of two matrices can be the zero matrix without either matrix being the zero matrix.

Problem 29 Write

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

as two column vectors

$$B = [B_1\ B_2] \quad \text{where} \quad B_1 = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix}$$

then solve the two systems

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} b_{21} \\ b_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Problem 30 Think about the fact that A is $n \times n$ and noninvertible and what that says about the number of solutions to the system

$$A\vec{x} = \vec{0}$$

has (Fact 2.3.4b). Can you think of a way to construct B from these solutions?

Problem 31 Write A as two column vectors as two column vectors

$$A = [A_1\ A_2] \quad \text{where} \quad A_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}$$

then find the general solution of the two systems

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} a_{21} \\ a_{22} \\ a_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Problem 32 B is not square, so it is not invertible. Think about the existence of a solution to the system

$$B\vec{x} = \vec{0}$$

other than the zero vector (Fact 2.3.4b). Multiply both sides of this equation on the right by A and think about whether the resulting equation can possibly hold.

Problem 33 If AB is invertible, consider the implications of fact 2.3.4b for the system

$$(AB)\vec{x} = \vec{0}$$

Now consider the fact that since B is not square, it is not invertible. What does fact 2.3.4 say about the system

$$B\vec{x} = \vec{0}$$

Problem 34 Use the hint in the book and the associativity of matrix multiplication to write

$$AB(AB)^{-1} = A(B(AB)^{-1}) \quad \text{and} \quad (AB)^{-1}(AB) = ((AB)^{-1}A)B$$

Use fact 2.4.9.

Problem 35a Multiply both sides on the left by B and use the fact that $AB = I_m$.

Problem 35b The system

$$B\vec{y} = \vec{b}$$

is consistent if it has a solution for any $\vec{y} \in \mathbb{R}^n$. What happens if $\vec{y} = A\vec{x}$?

Problem 36 Write

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

as two column vectors

$$X = [X_1\ X_2] \quad \text{where} \quad X_1 = \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} \quad \text{and} \quad X_2 = \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix}$$

then solve the two systems

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Problem 37 Multiply both sided of the equation on the right by S. Now let S be an arbitrary 2×2 matrix

$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and equate the results of the matrix multiplications on each side element by element to get 4 equations.

Problem 38 Same hint as problem 37.

Problem 39 Let

$$X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where A is arbitrary. Equate the matrix products

$$XA \; = \; AX$$

element by element and solve the resulting four equations.

Supplementary Problem 1. Use the fact that $T(\vec{x}) = A\vec{x}$ and fact 1.3.9 (p.32).

Supplementary Problem 2. Suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Write

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

as a sum of products of the *scalars* x_1, x_2 with the *standard vectors* (unit vectors in the direction of the coordinate axes) in $\mathbb{R}^2$, $\vec{e}_1, \vec{e}_2$:

$$\vec{x} \; = \; \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \; = \; x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For each i, $1 \leq i \leq m$,

$$T(x_i \vec{e}_i) \; = \; x_i T(\vec{e}_i) \in \mathbb{R}^n$$

Think of how to arrange these vectors to form A.