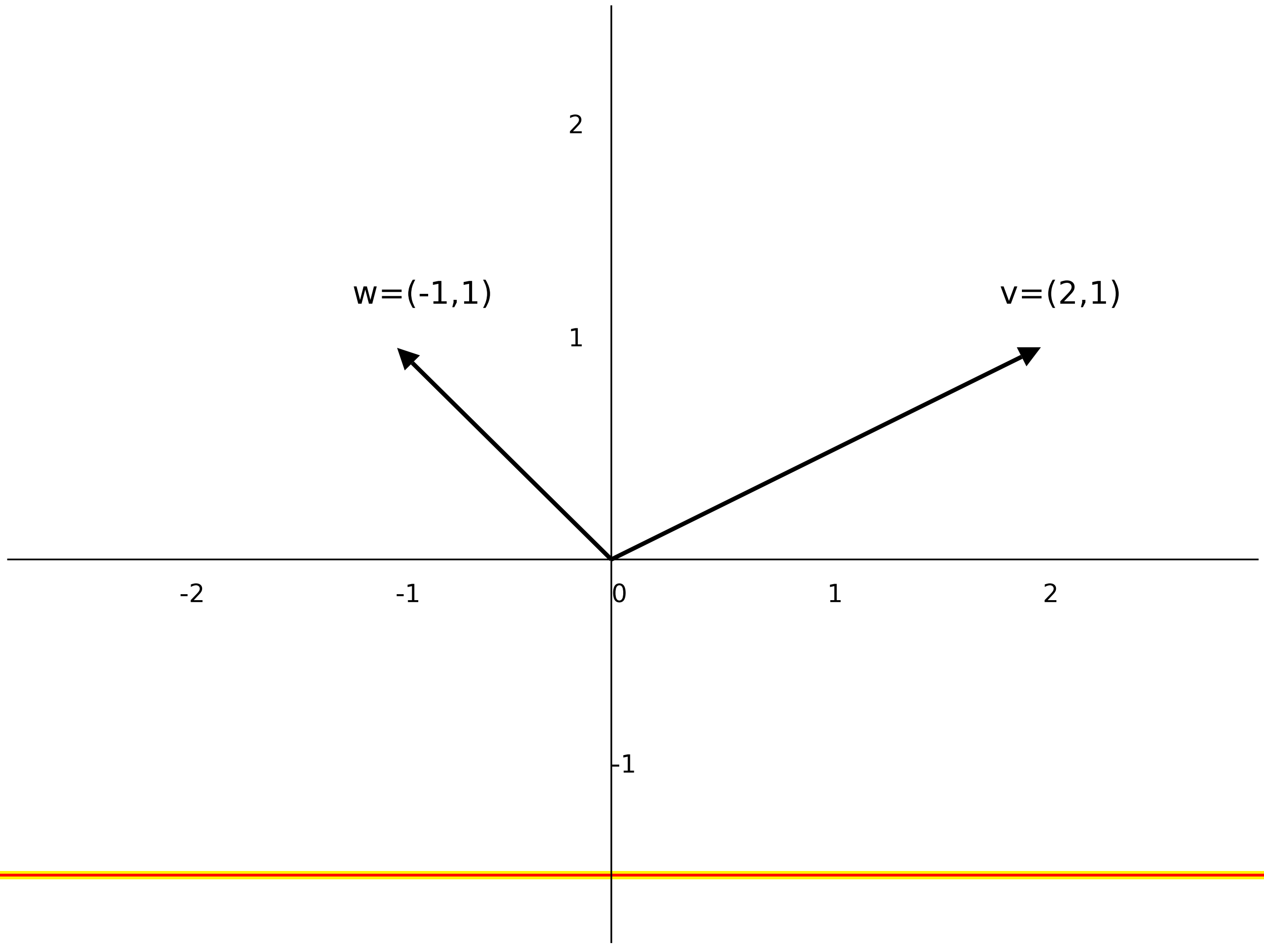
Vector Algebra

Gene Quinn

Graphical Representation

The *standard representation* of a vector $\vec{v} \in \mathbb{R}^2$ is a line from the origin to the point with coordinates equal to the components of $\vec{v}$, namely (v_1, v_2), with an arrowhead on the end at (v_1, v_2):



Vector Algebra

The first operation we will define is the product of a scalar $k \in \mathbb{R}$ times a vector $\vec{v} \in R^n$.

There are a number of ways this product could be defined, but the usual choice is:

$$k\vec{v} = k \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} kv_1 \\ kv_2 \\ \vdots \\ kv_n \end{bmatrix}$$

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So $k\vec{v}$ is simply the original vector $\vec{v}$ with each component multiplied (using ordinary real number multiplication) by k.

Vector Algebra

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So, the following vectors are parallel:

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 9 \\ 18 \end{bmatrix}$$

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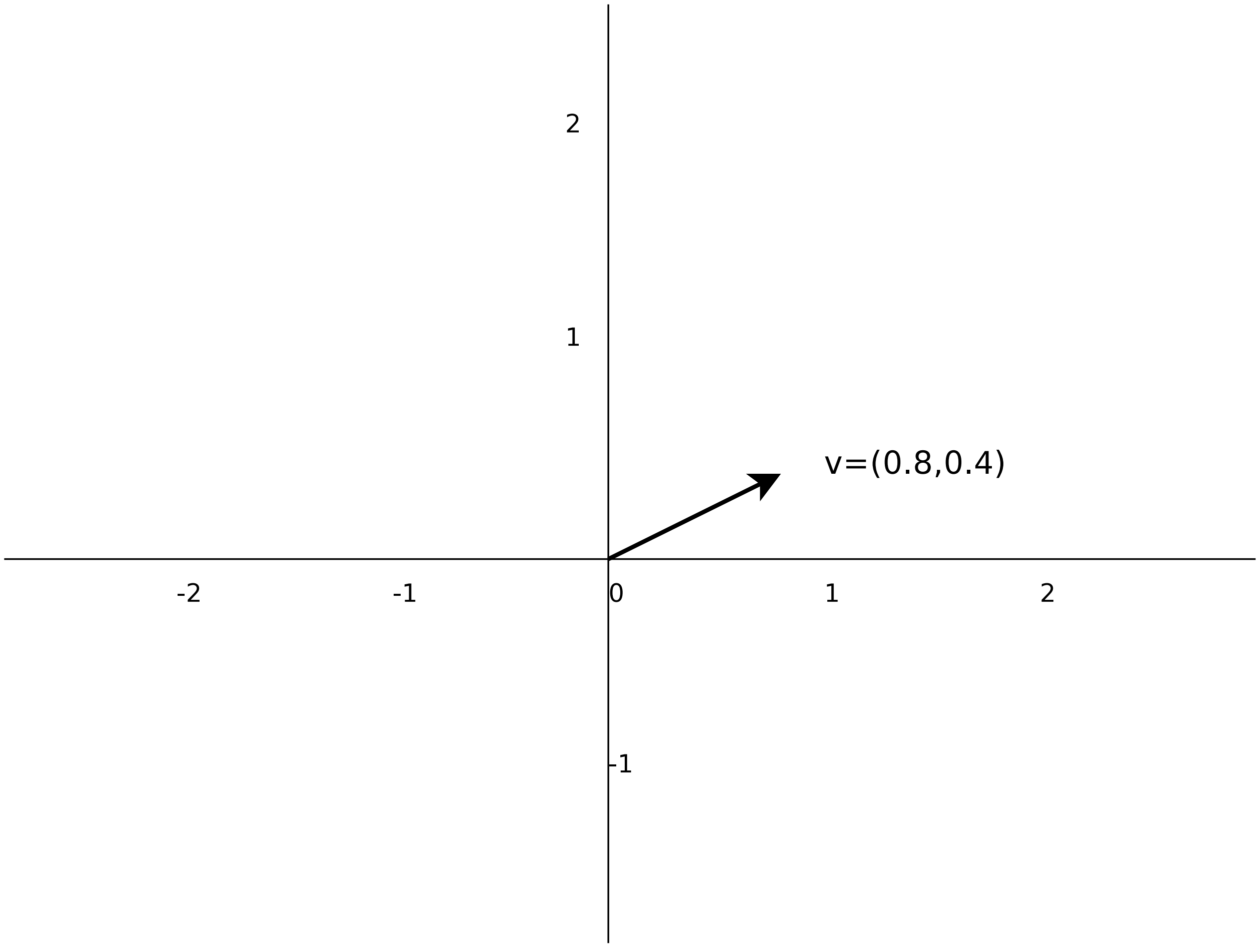
The set of all vectors parallel to a given vector $\vec{v}$ can be depicted as a line through the origin having the same direction as $\vec{v}$.

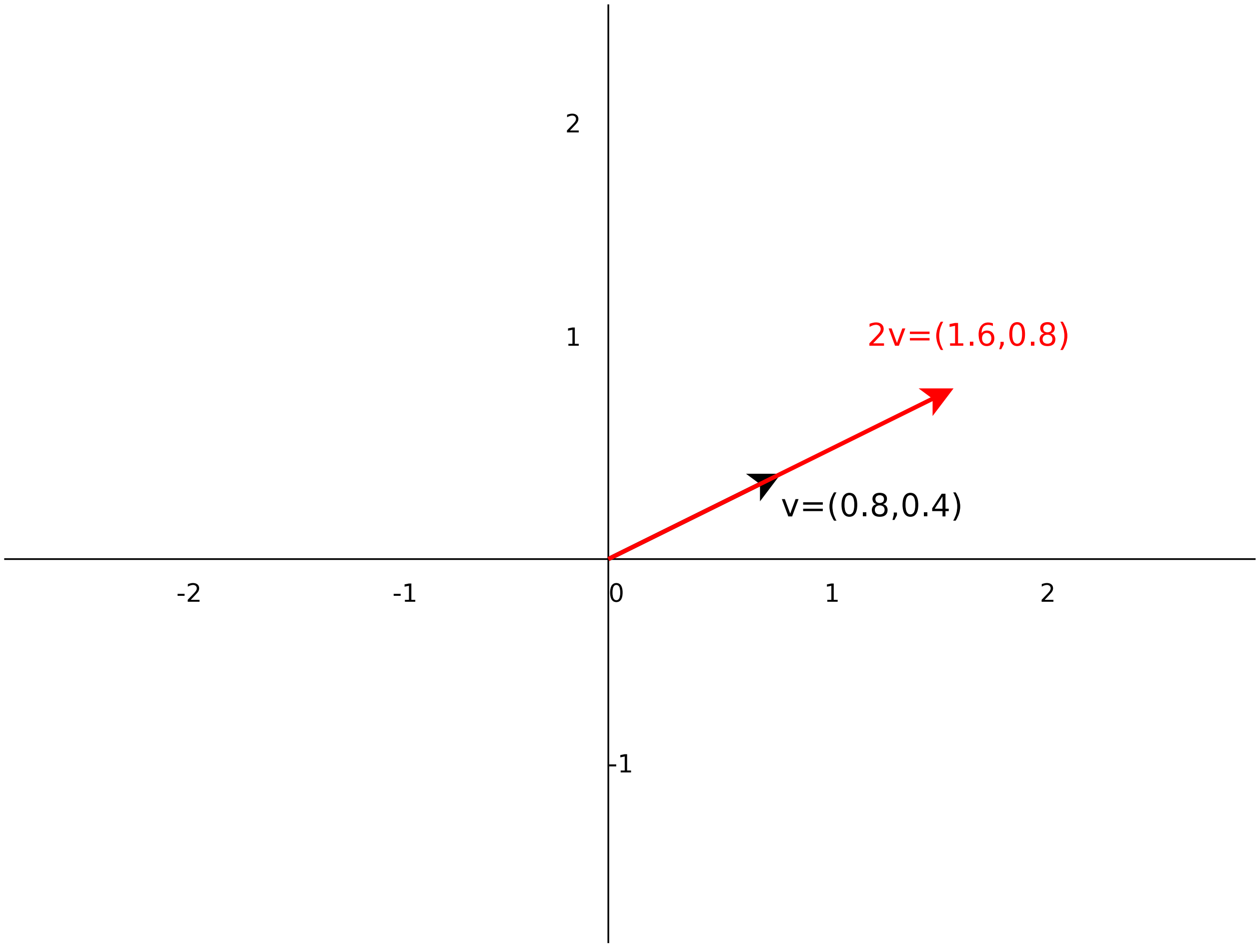
Multiplication by a Scalar

We start with the vector $\vec{v} = (0.8, 0.4)$, which is represented graphically as:



Multiplication by a Scalar

The vector $2\vec{v} = (2 \cdot 0.8\,, 2 \cdot 0.4) = (1.6, 0.8)$ is represented graphically as:



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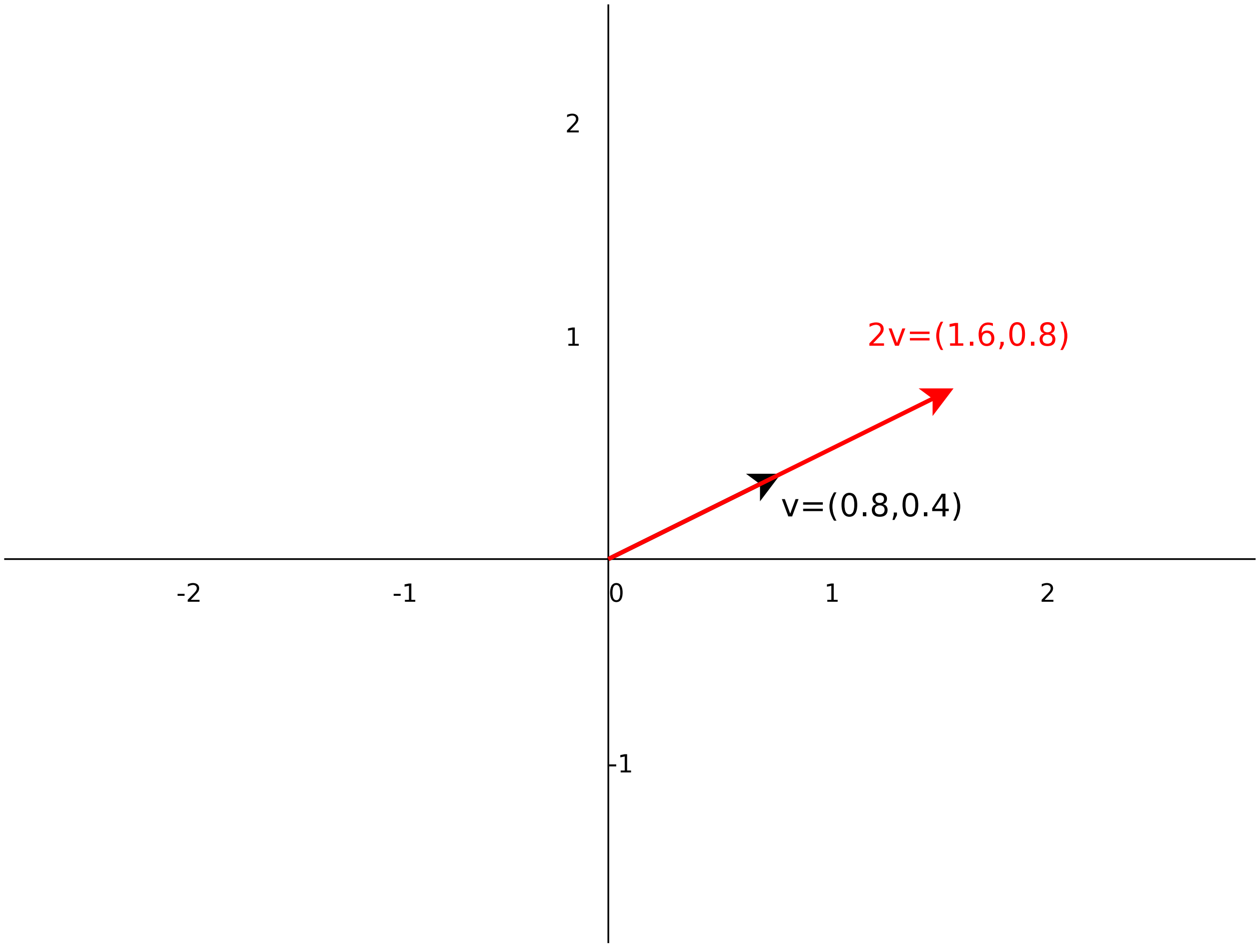
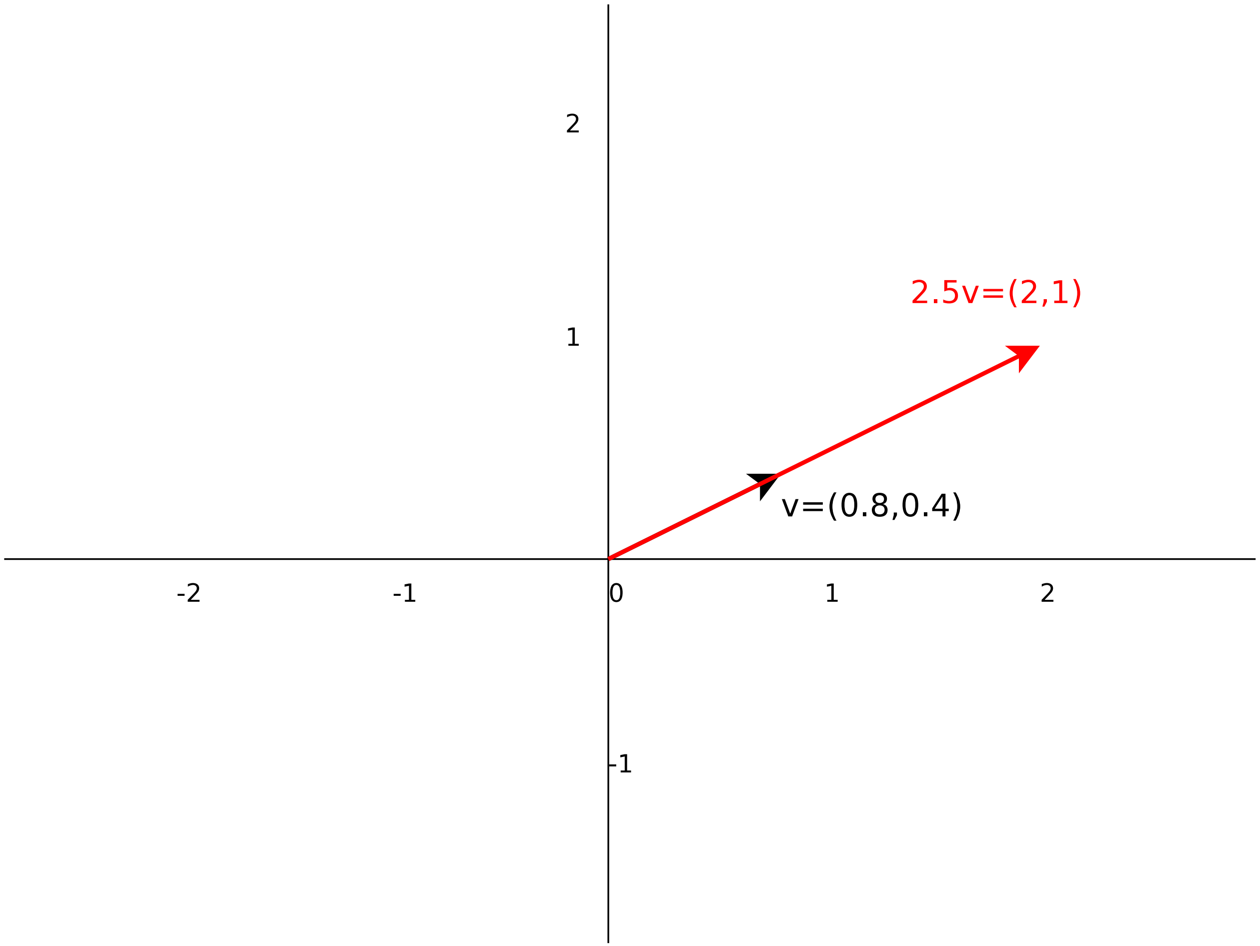


Note that $2\vec{v}$ is parallel to $\vec{v}$.

Multiplication by a Scalar

The vector $2.5\vec{v} = (2.5 \cdot 0.8, 2.5 \cdot 0.4) = (2, 1)$ is represented graphically as:



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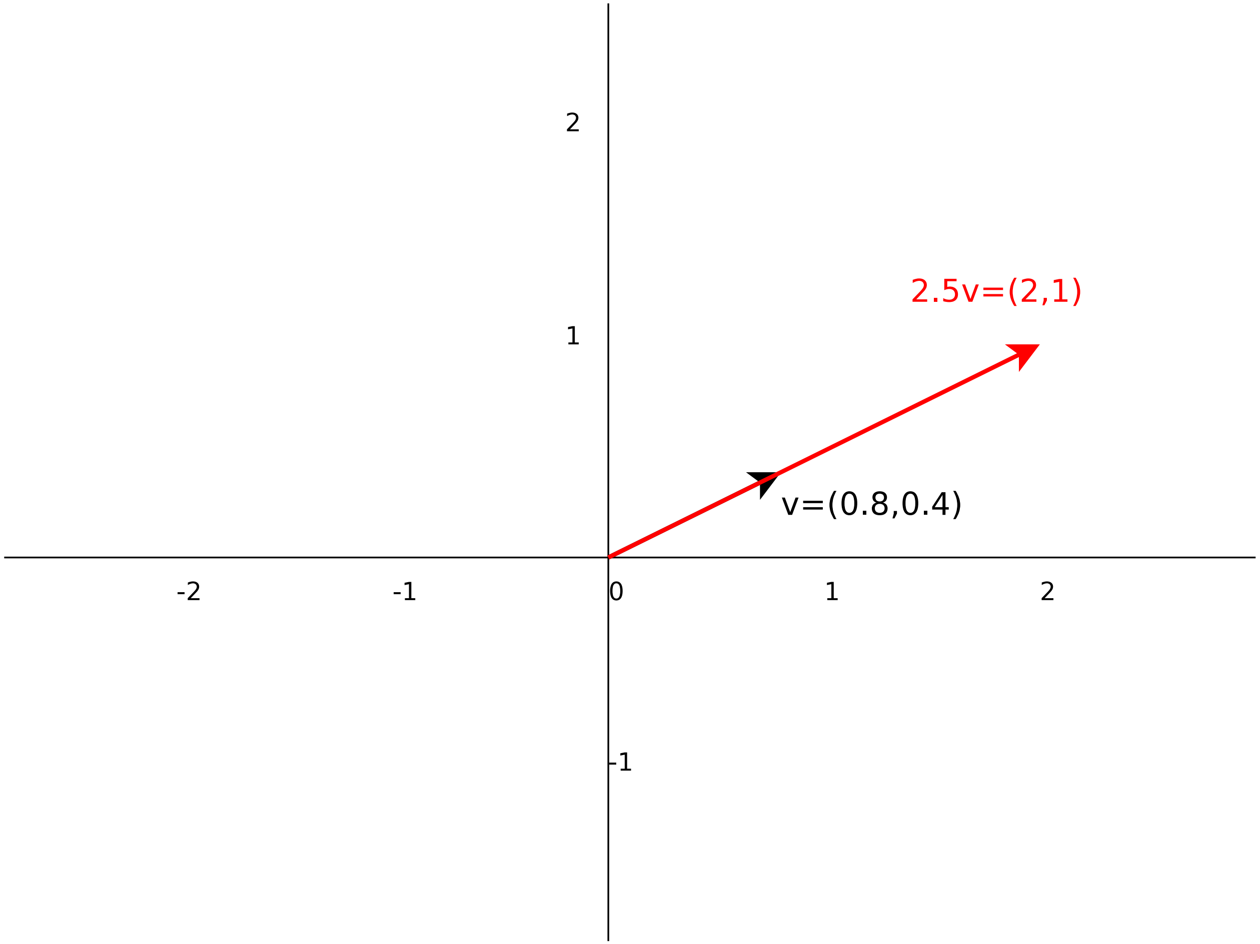
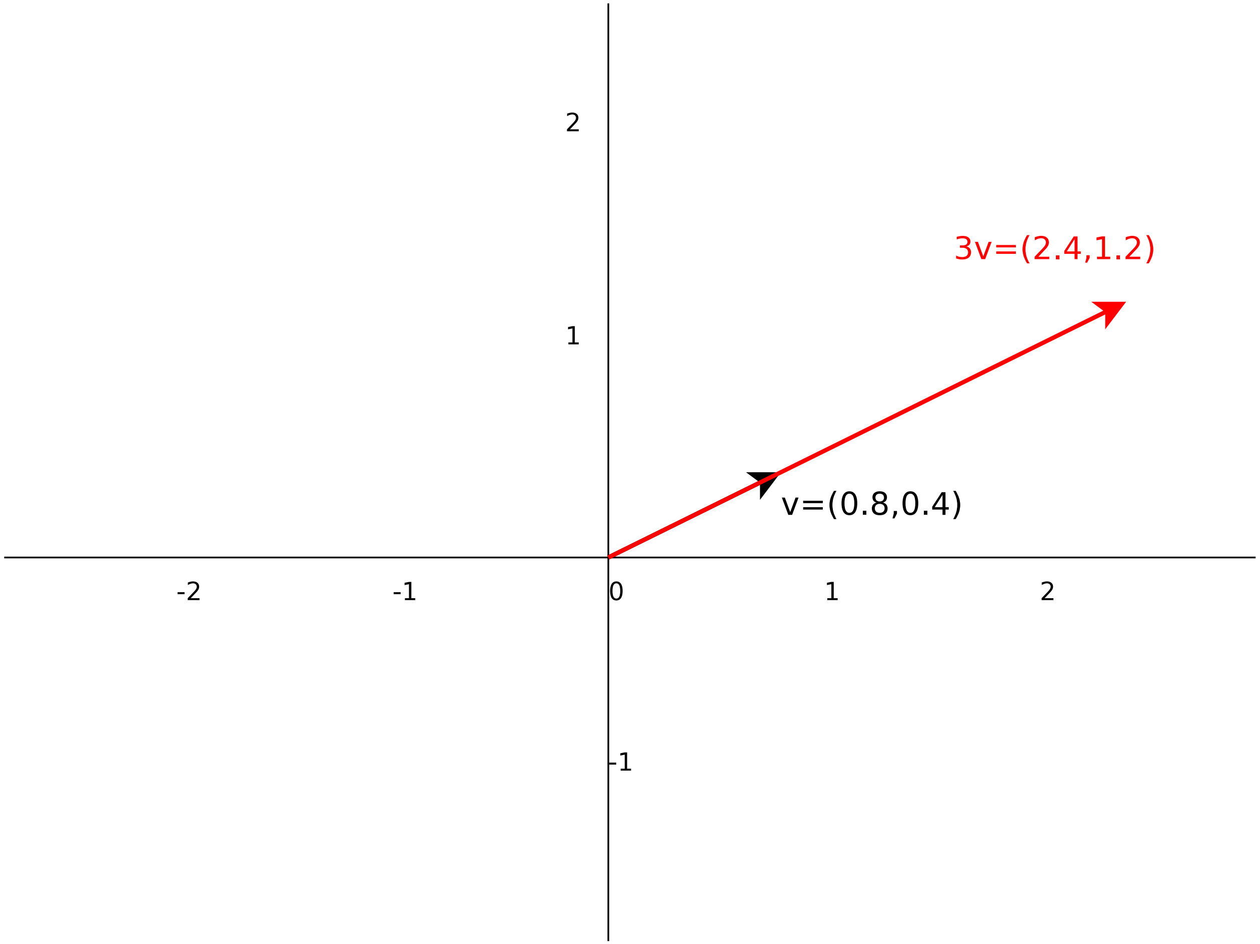


Note that $2.5\vec{v}$ is parallel to $\vec{v}$.

Multiplication by a Scalar

The vector $3\vec{v} = (3 \cdot 0.8, 3 \cdot 0.4) = (2.4, 1.2)$ is represented graphically as:



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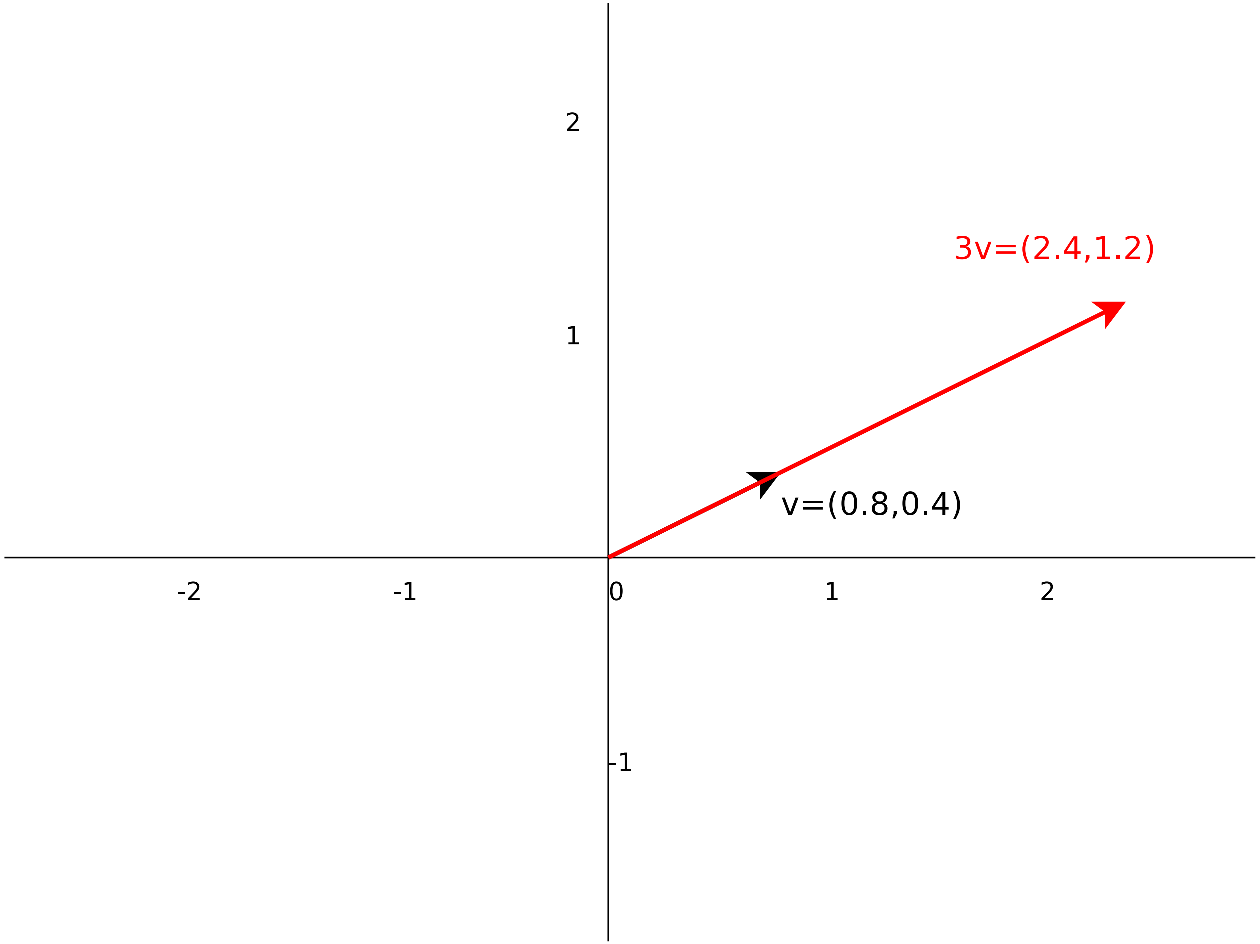
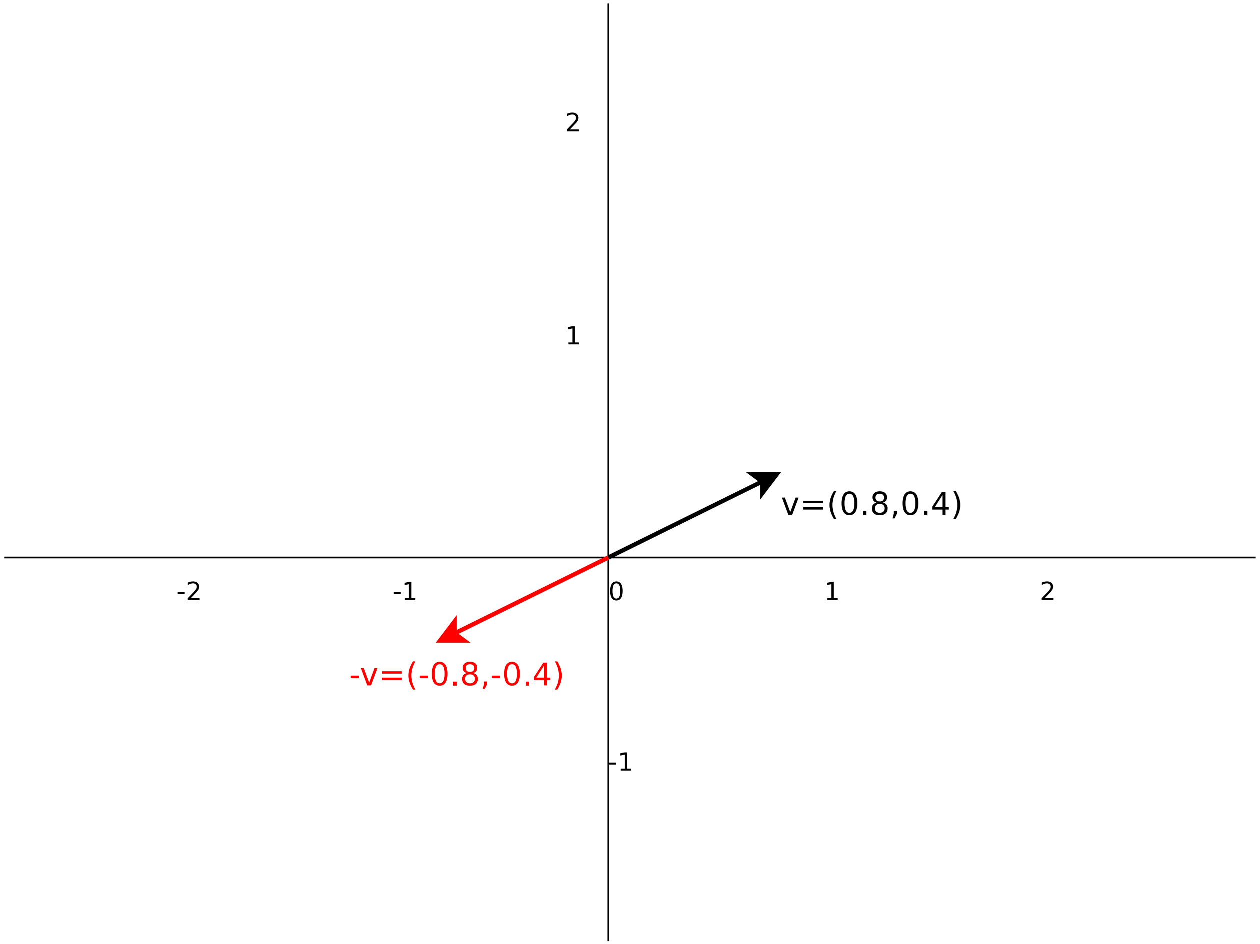


As before, $3\vec{v}$ is parallel to $\vec{v}$.

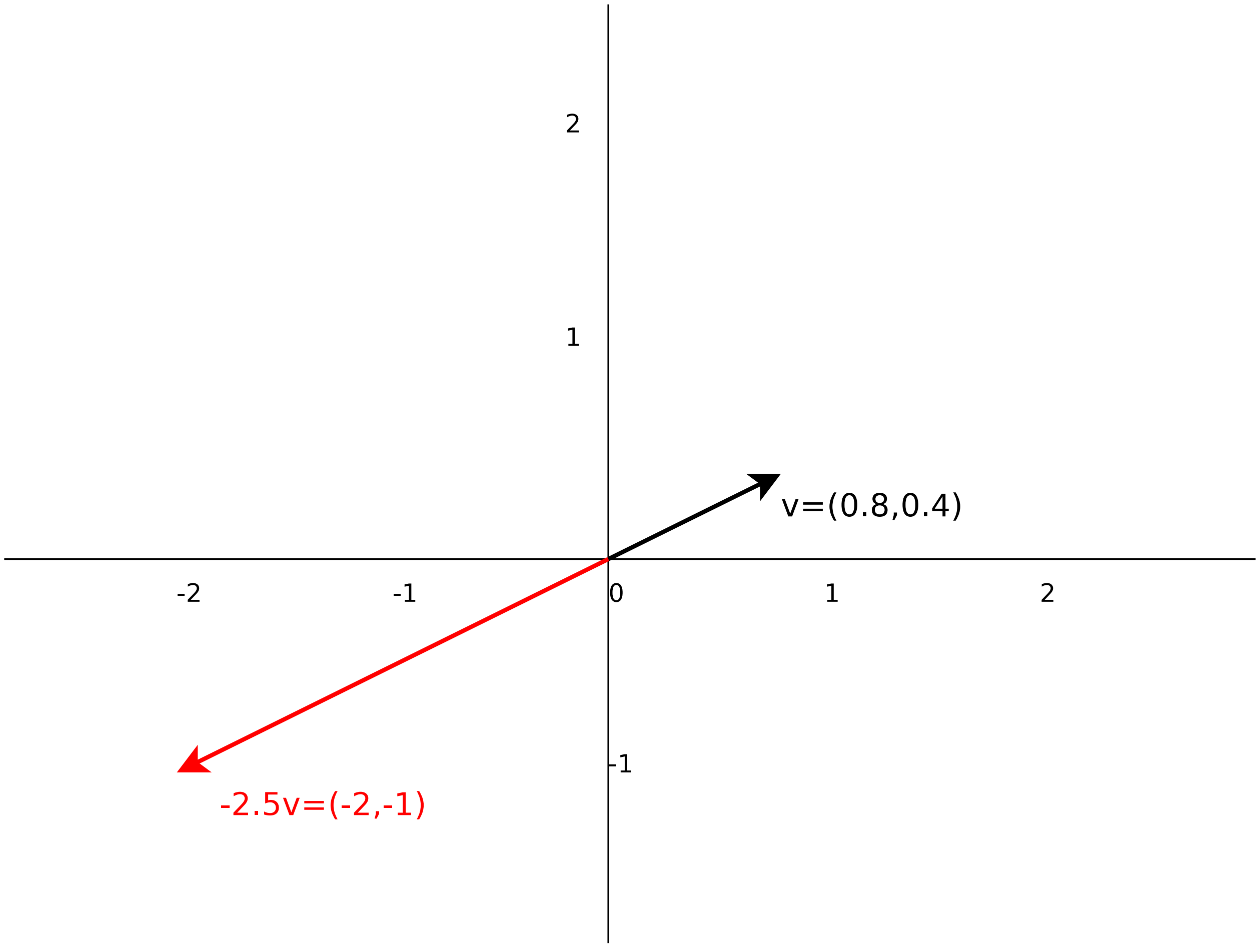
Multiplication by a Scalar

The vector $-\vec{v} = (-1 \cdot 0.8, -1 \cdot 0.4) = (-0.8, -0.4)$ is represented graphically as:



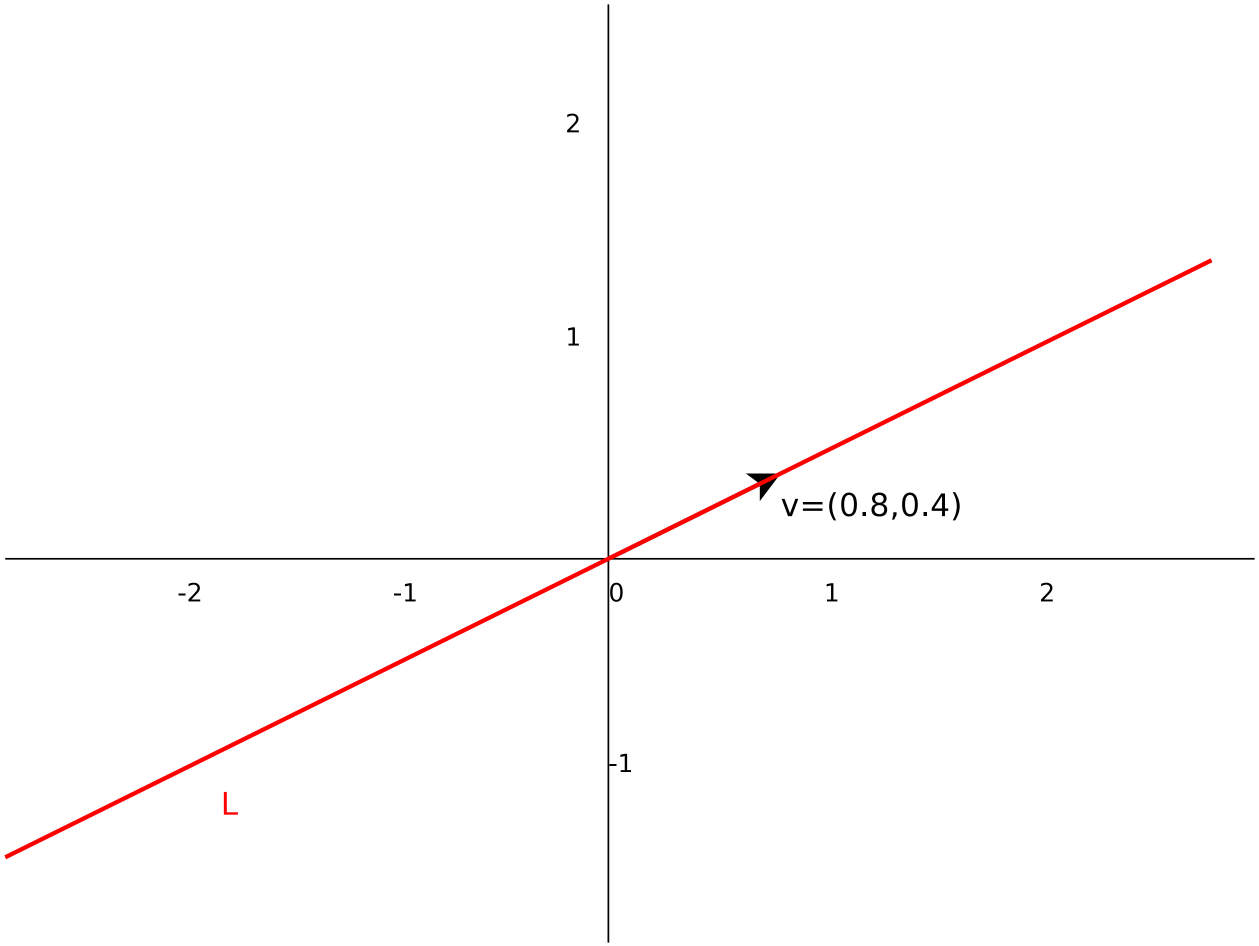
Multiplication by a Scalar

The vector $-2.5\vec{v} = (-2.5 \cdot 0.8, -2.5 \cdot 0.4) = (-2, -1)$ is represented graphically as:



Multiplication by a Scalar

The endpoints of the set of all vectors $W = \{\vec{w} \ : \ w = k\vec{v}, \quad k \in \mathbb{R}\}$ is represented graphically as the line L:



Vector Addition

The other operation we will define is the *sum* of two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ ($\vec{u}$ and $\vec{w}$ must have the same number of components).

$$\vec{v} + \vec{w} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix}$$

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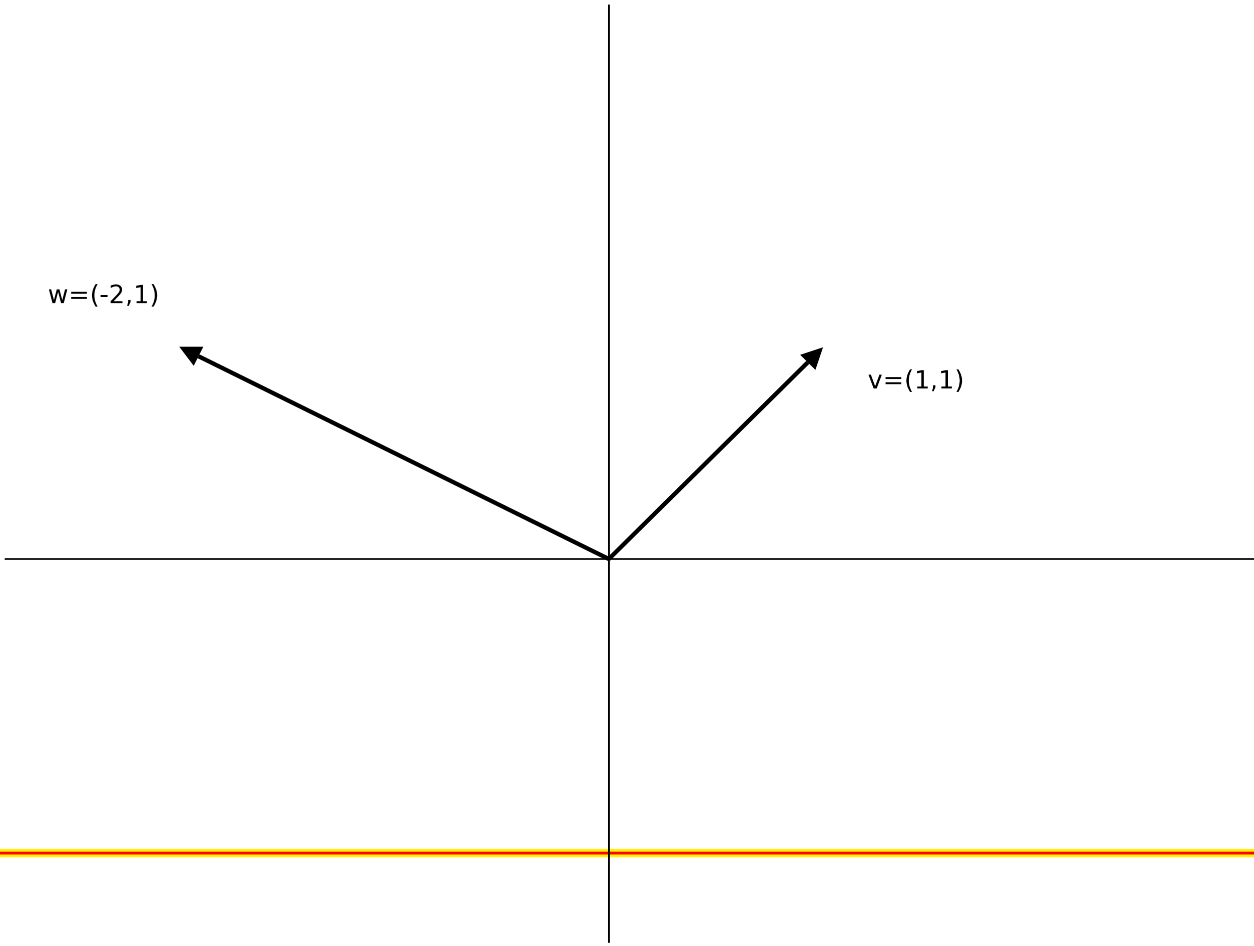
So $\vec{v} + \vec{w}$ is simply the vector that has its i^{th} component equal to the sum of the i^{th} components of $\vec{u}$ and $\vec{w}$.

Vector Addition

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{w} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$



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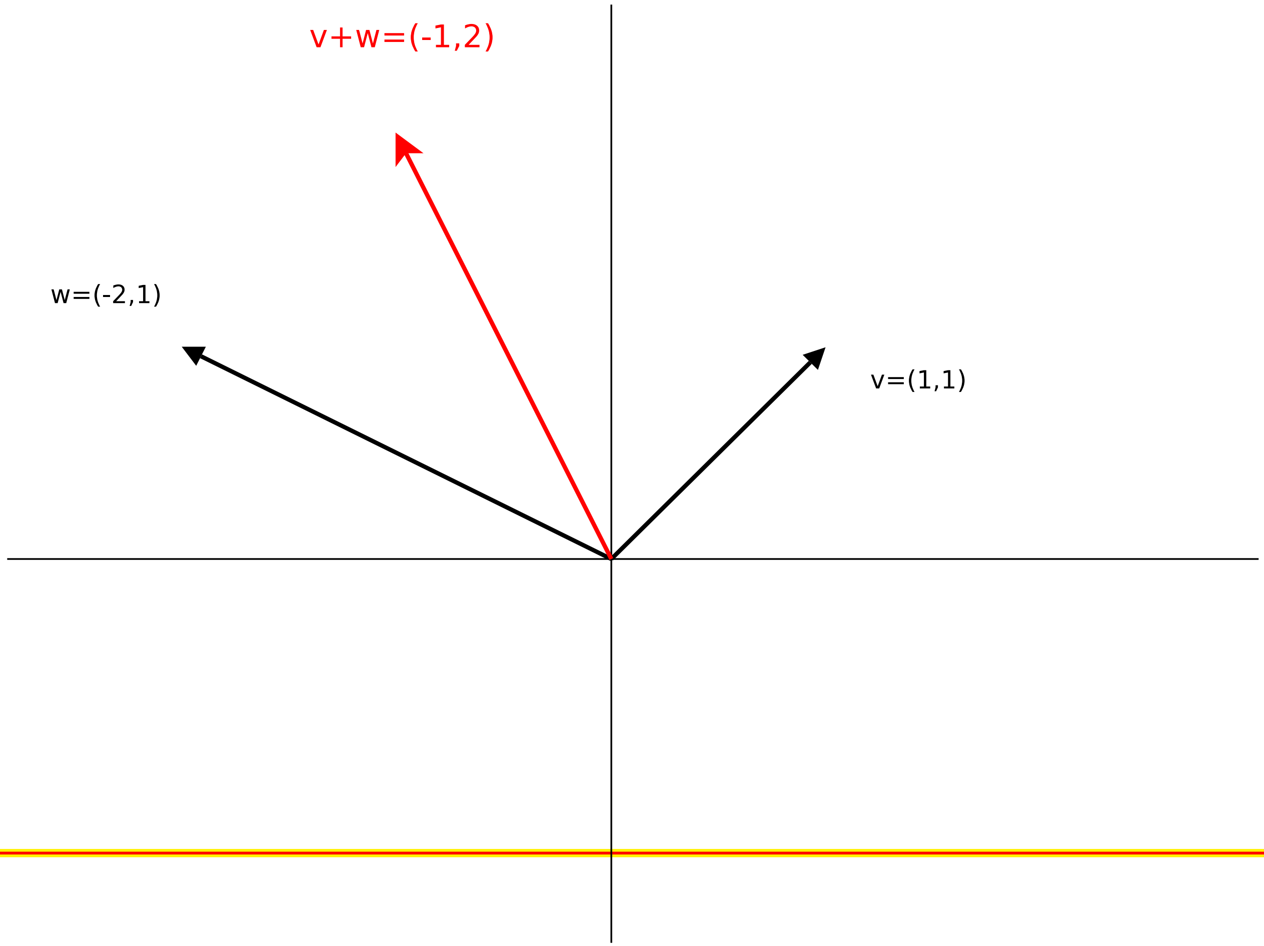

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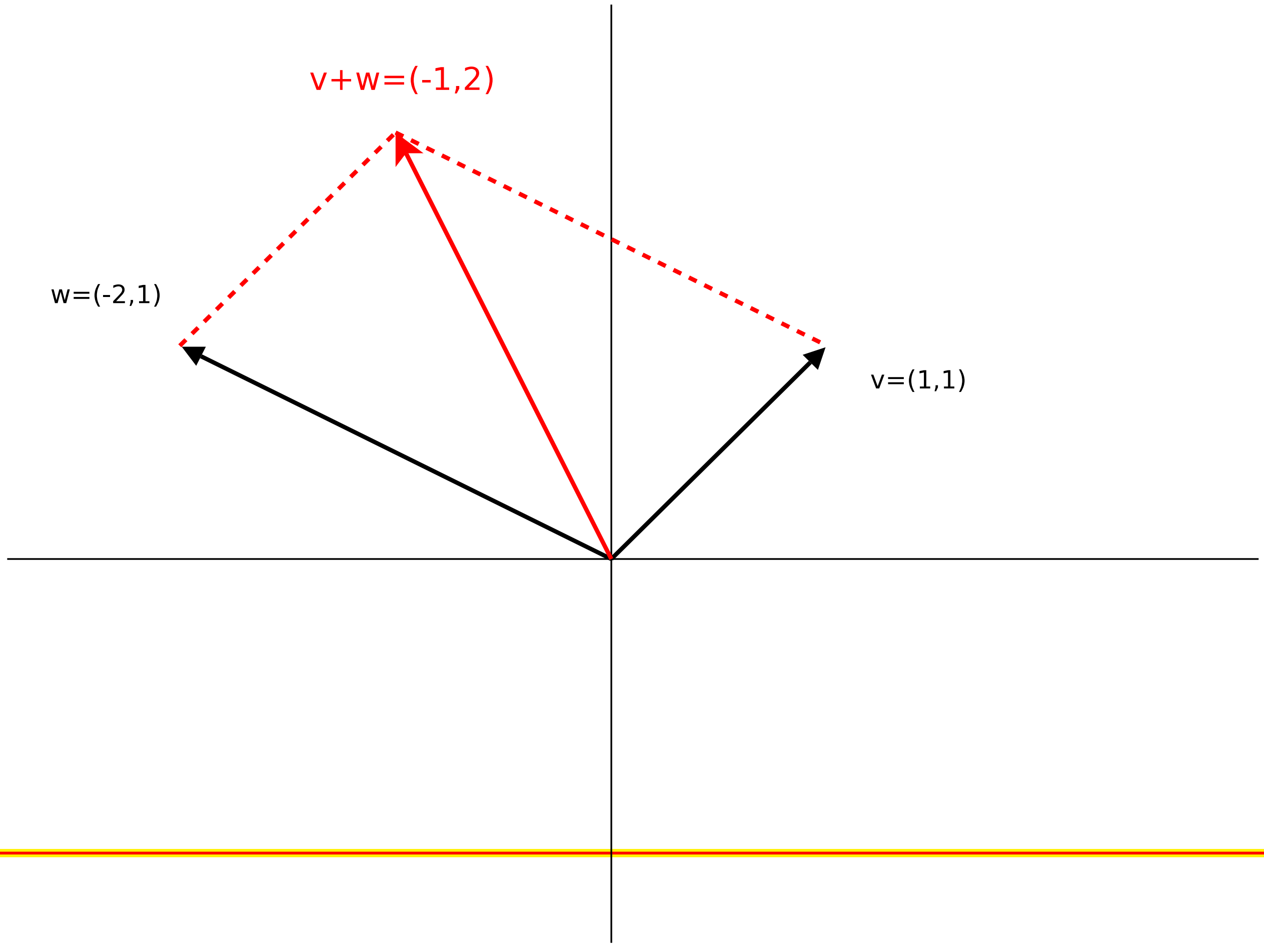

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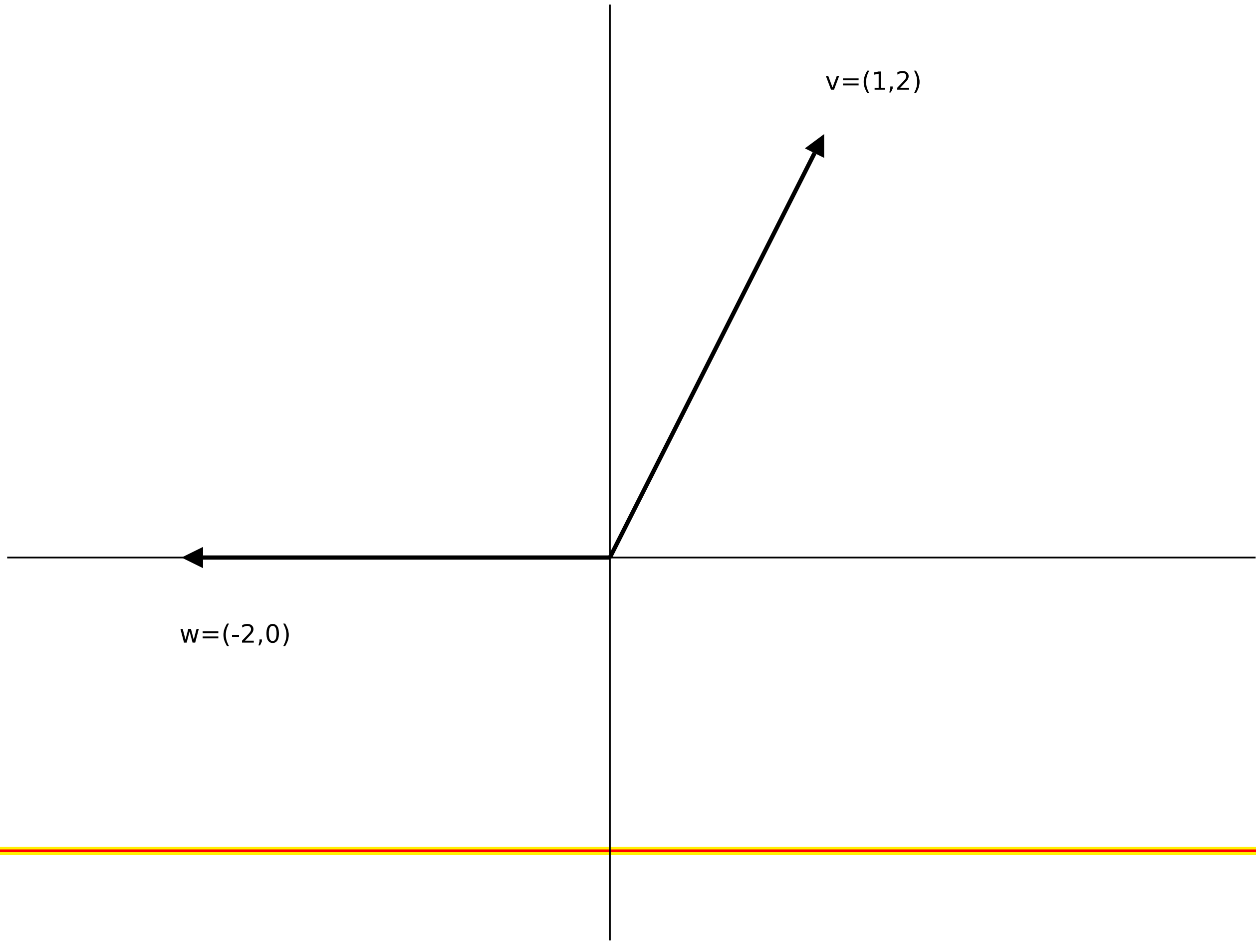

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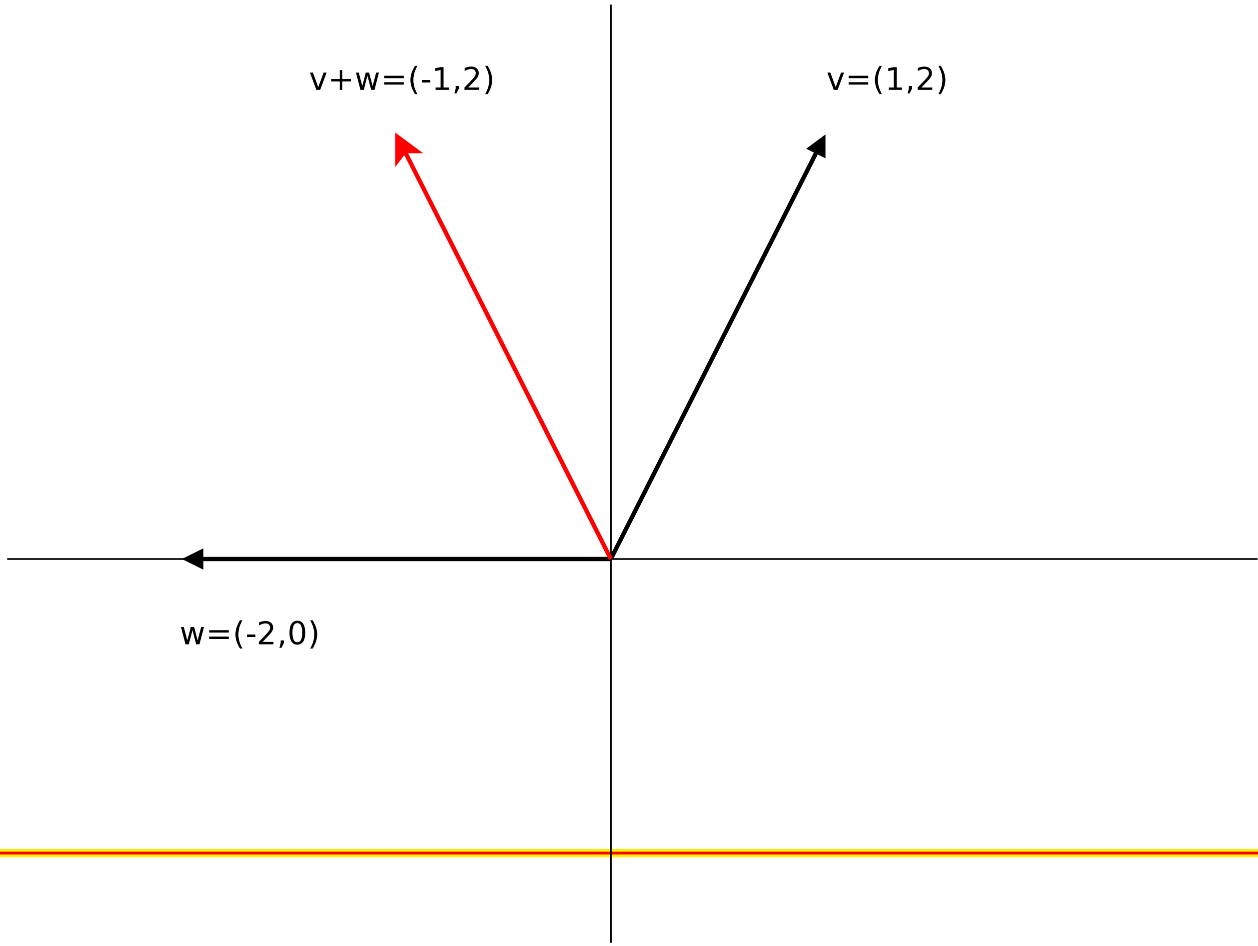

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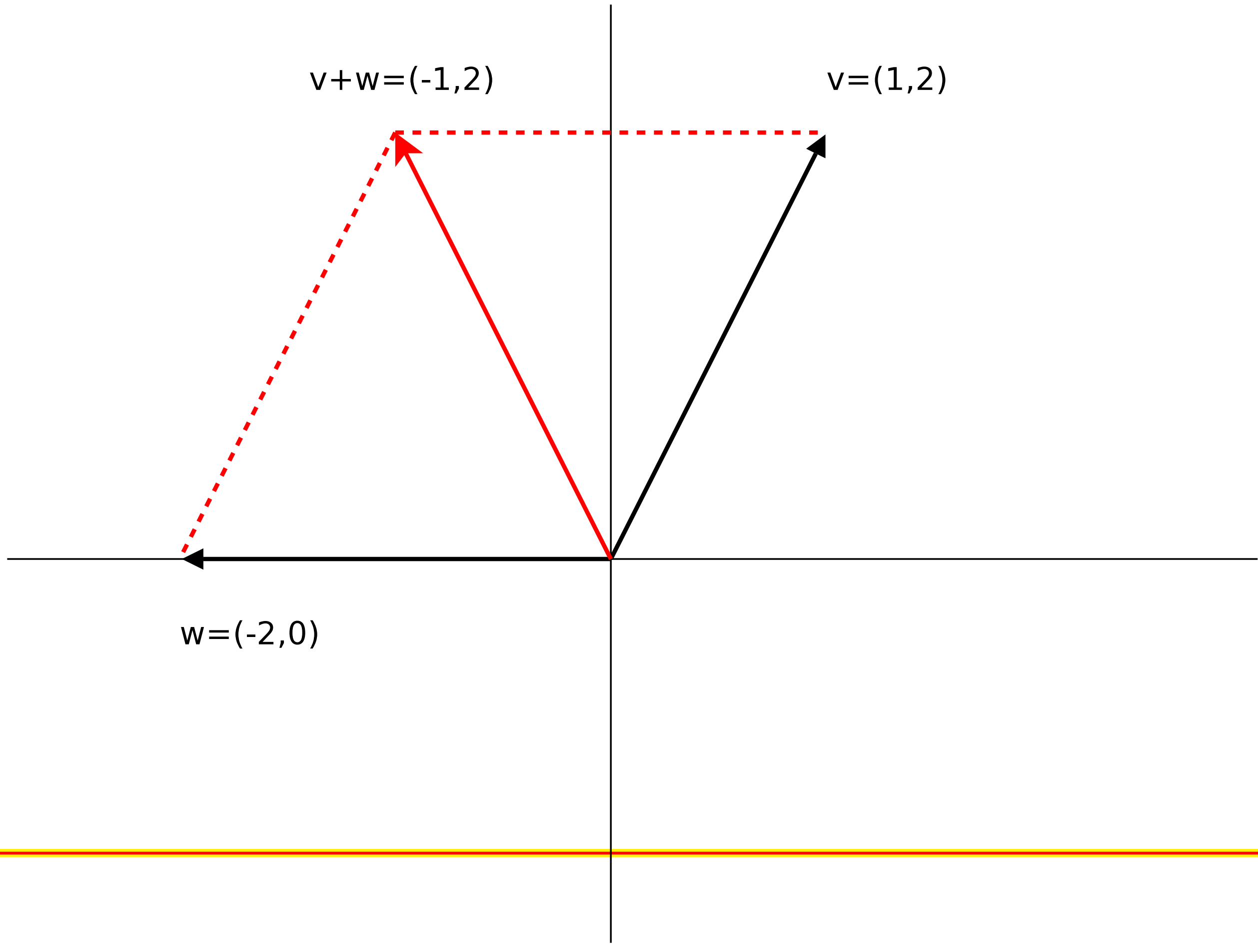

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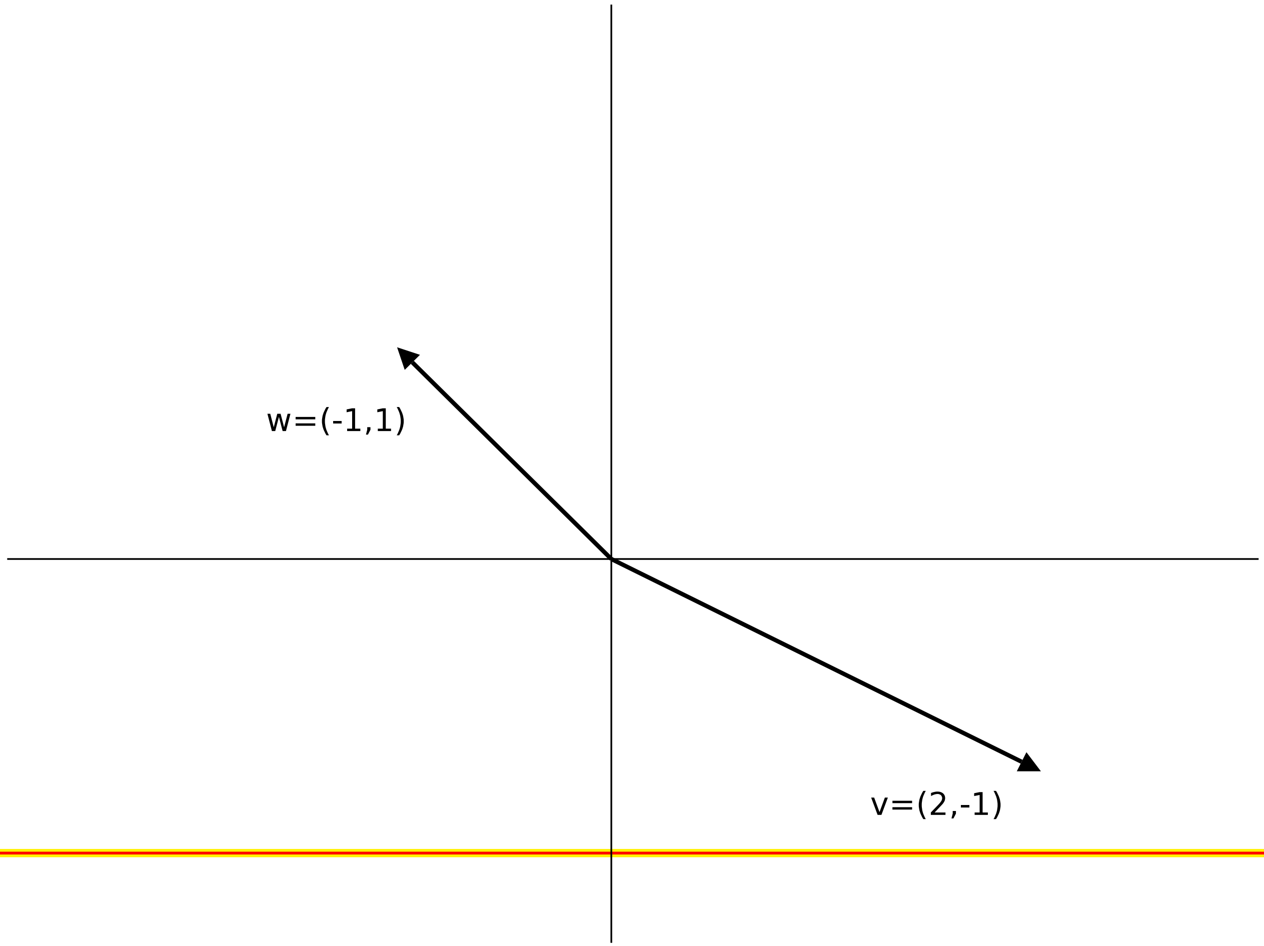

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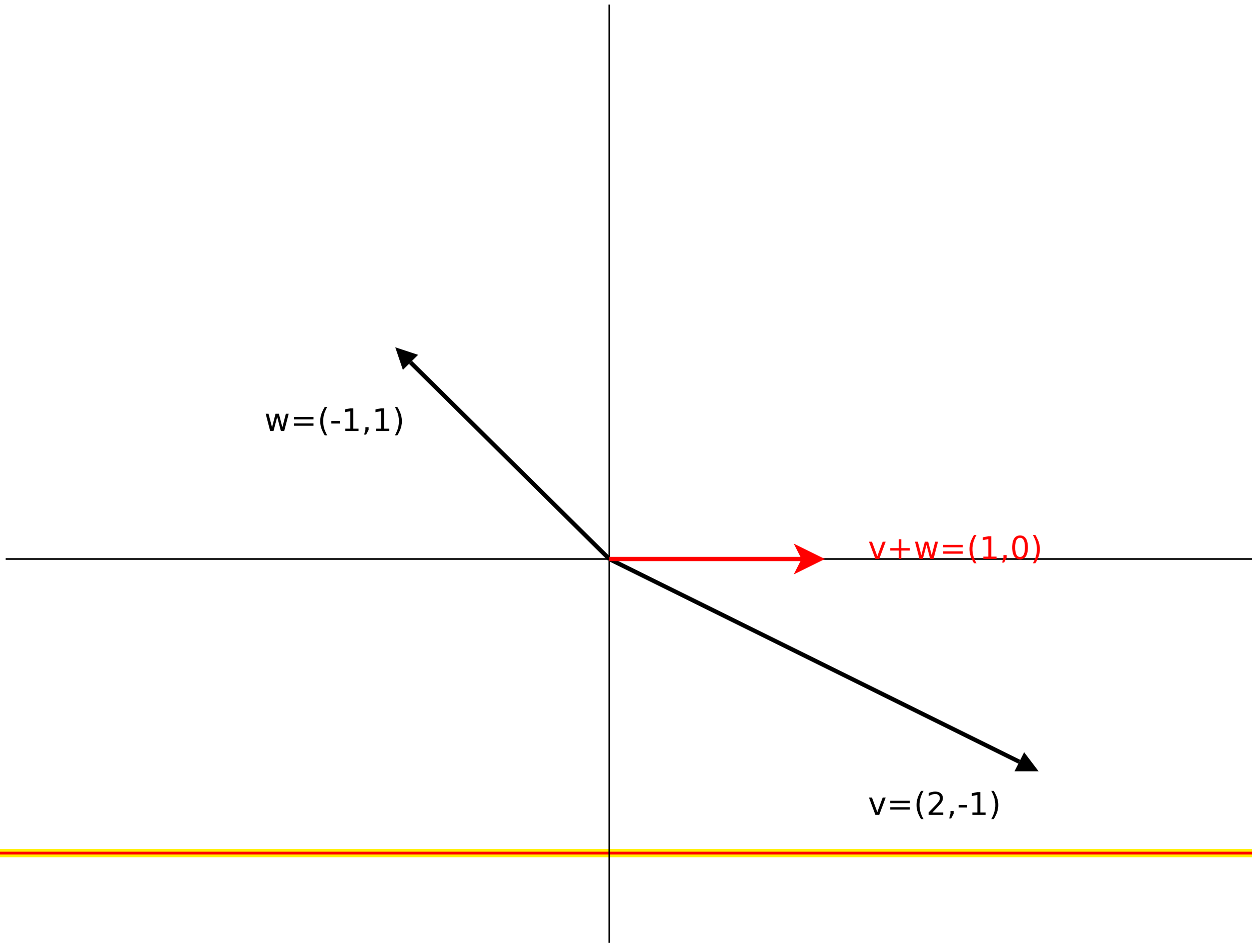

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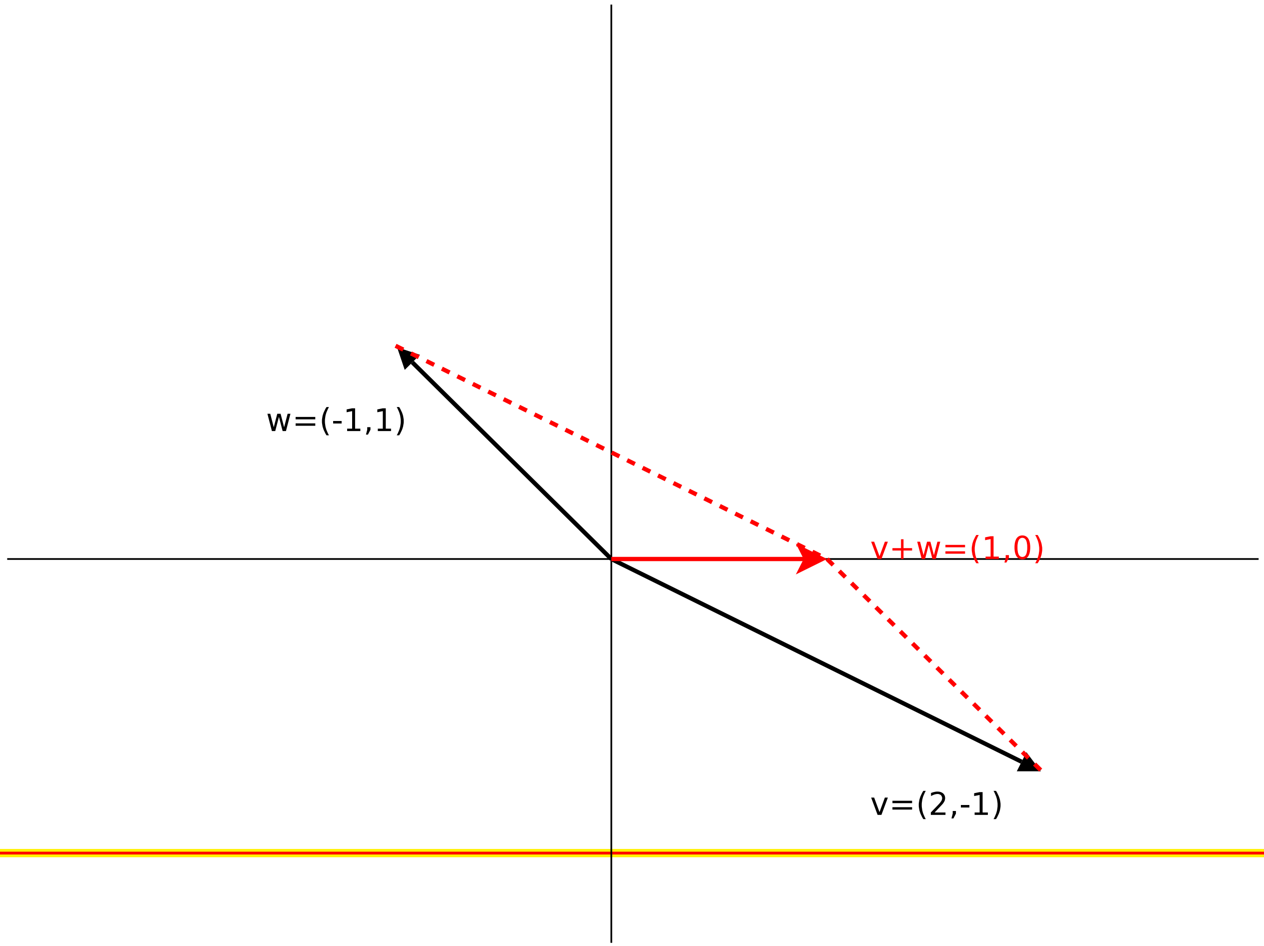
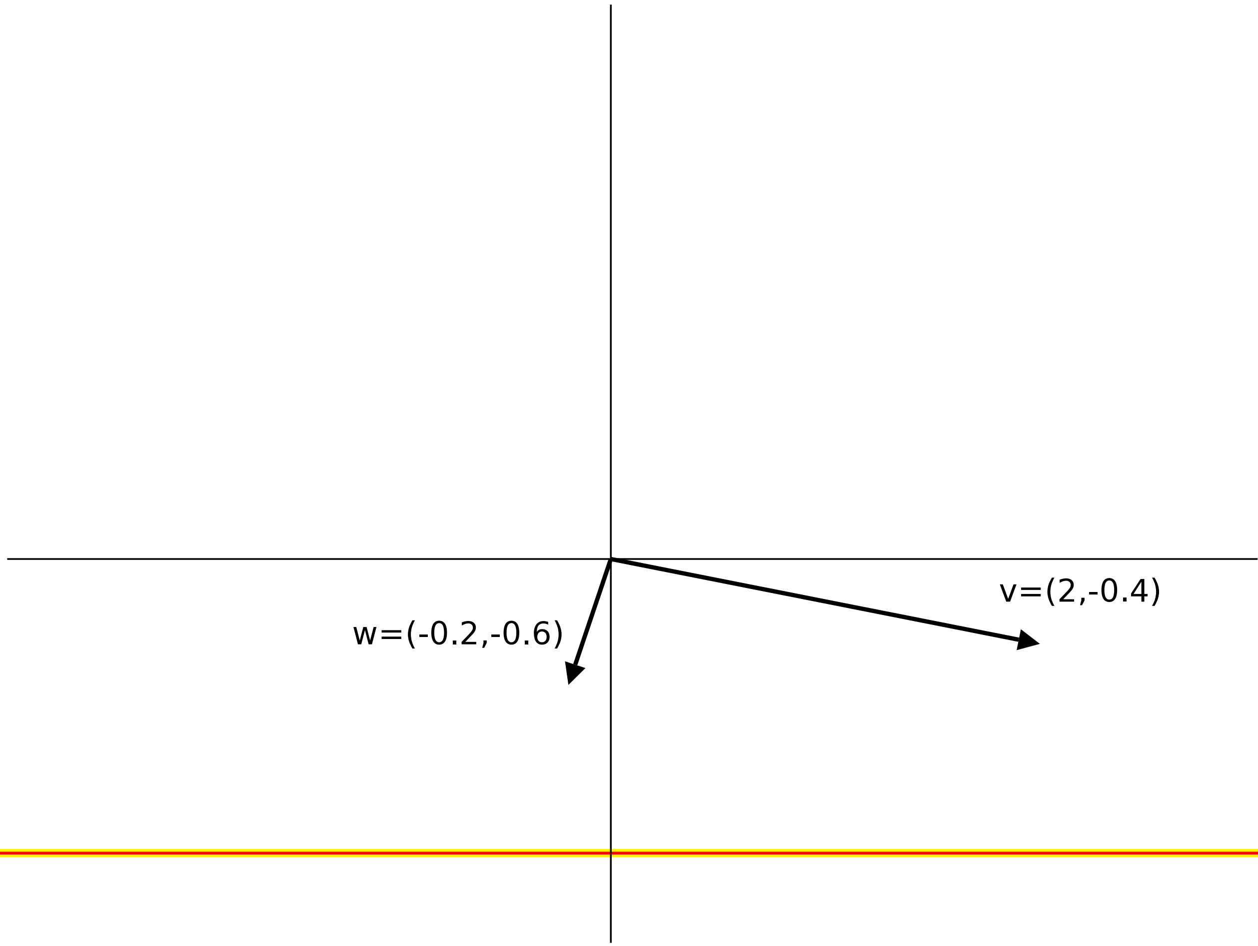


Vector Addition

$$\vec{v} = \begin{bmatrix} 2 \\ -0.4 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} -0.2 \\ -0.6 \end{bmatrix}$$



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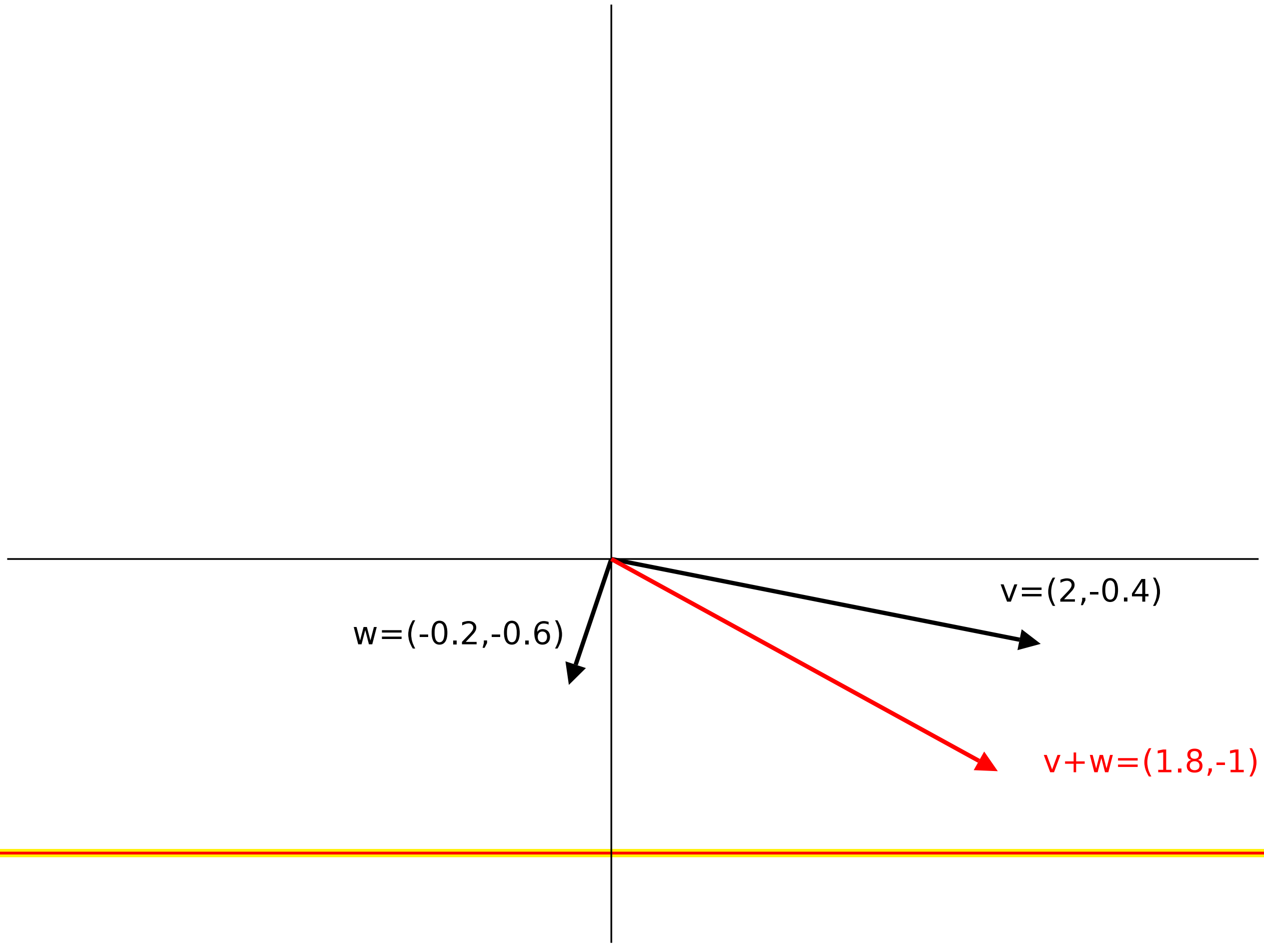
$$\vec{v} = \begin{bmatrix} 2 \\ -0.4 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} -0.2 \\ -0.6 \end{bmatrix} \quad \vec{v} + \vec{w} = \begin{bmatrix} 1.8 \\ -1 \end{bmatrix}$$



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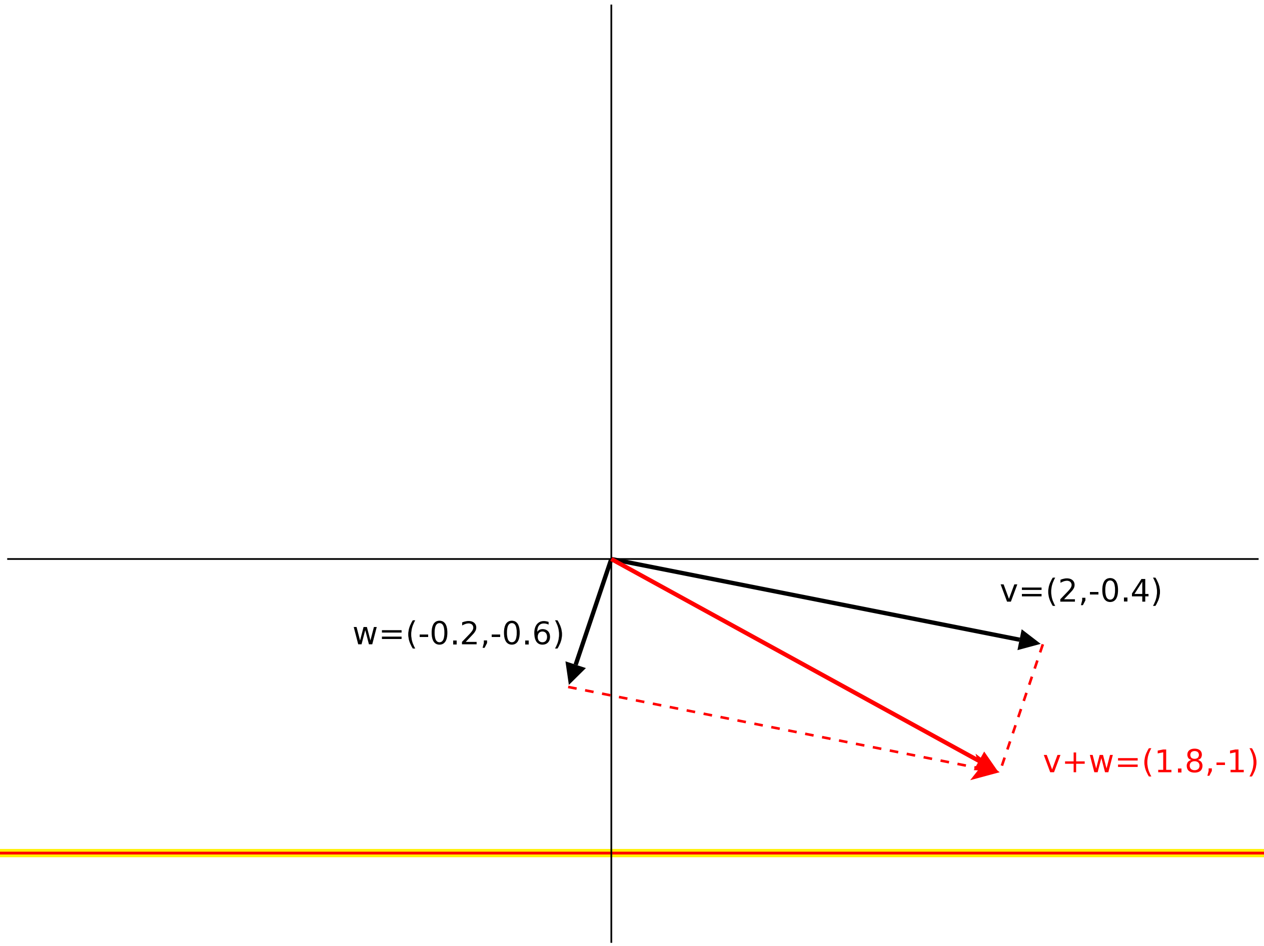
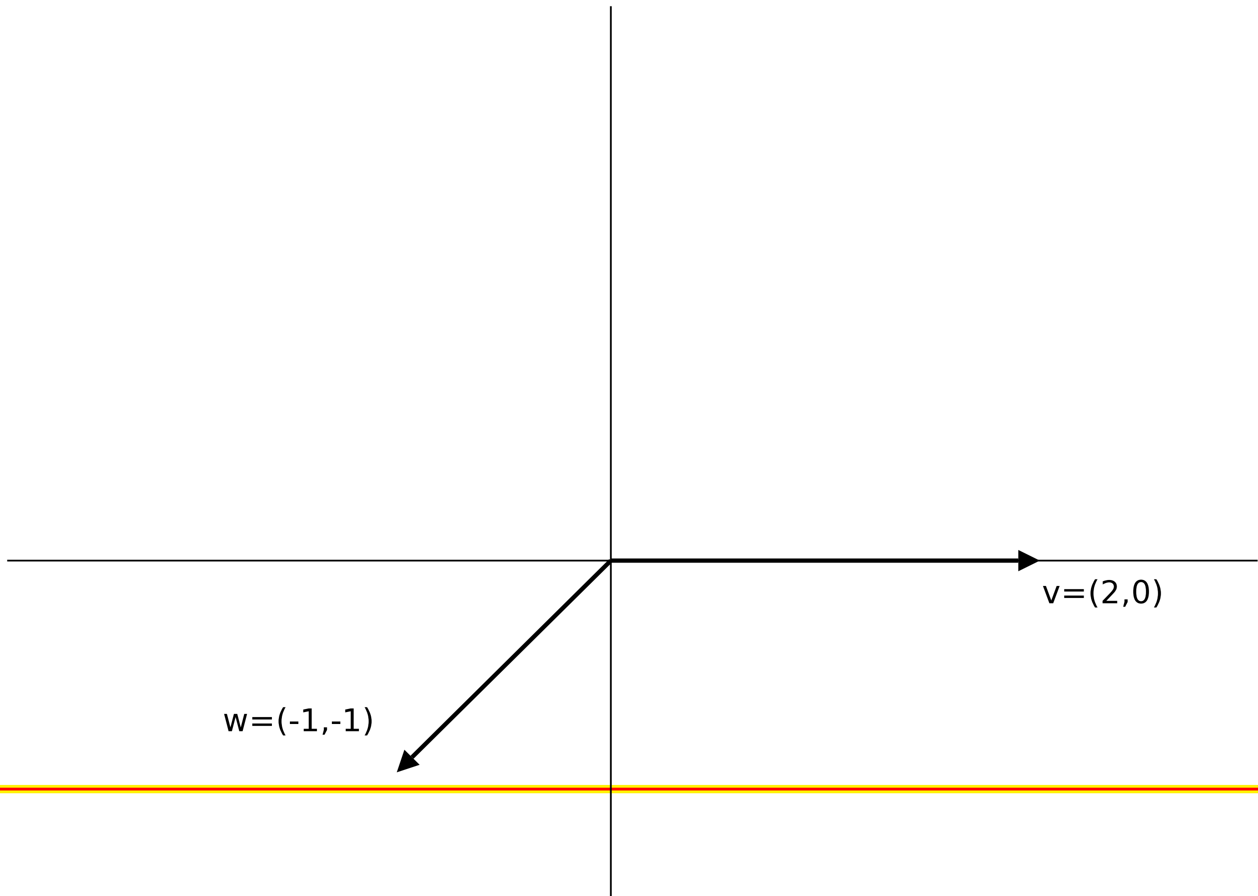


Vector Subtraction

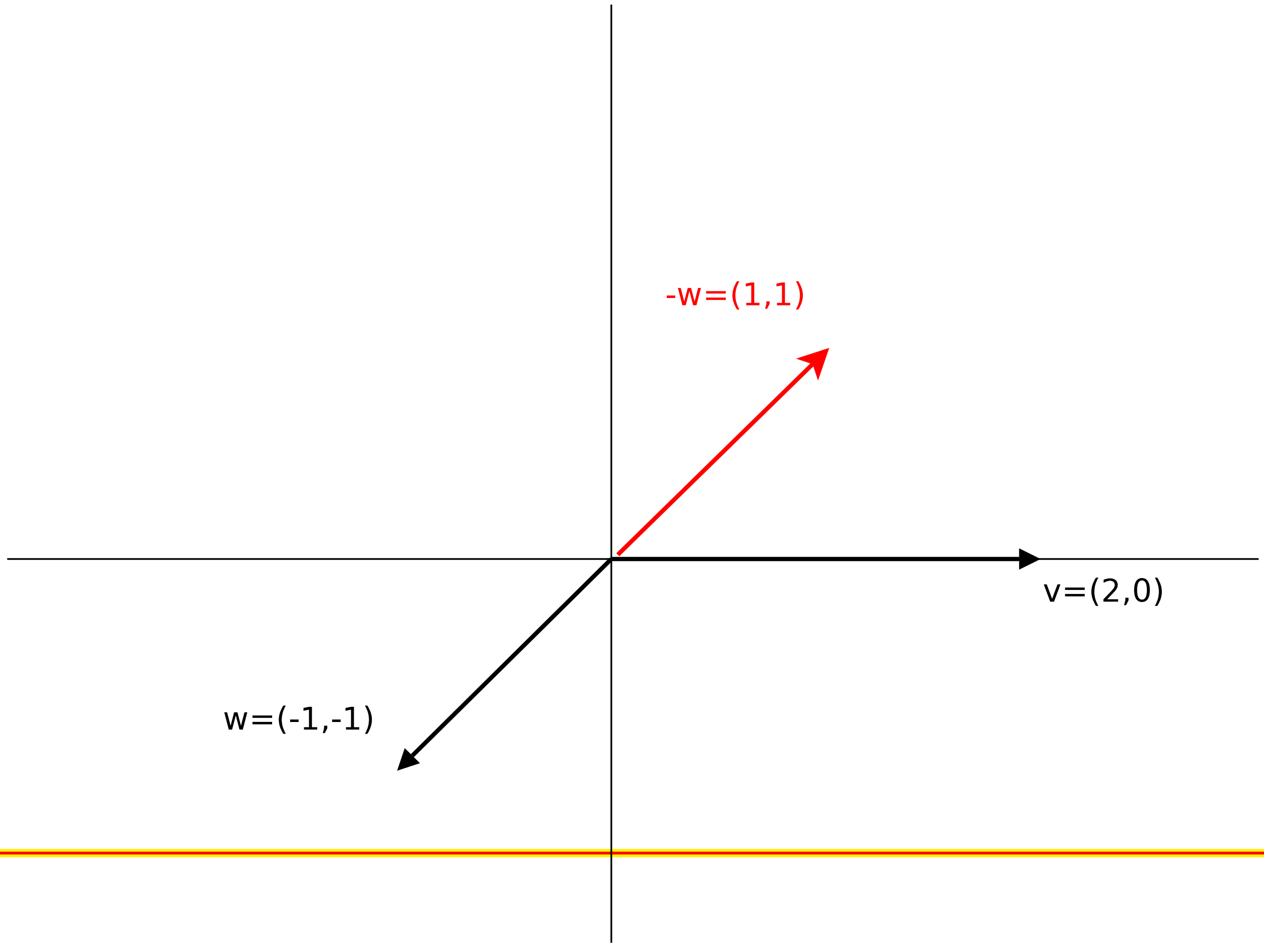
Subtraction is equivalent to negation followed by addition.

$$\vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$



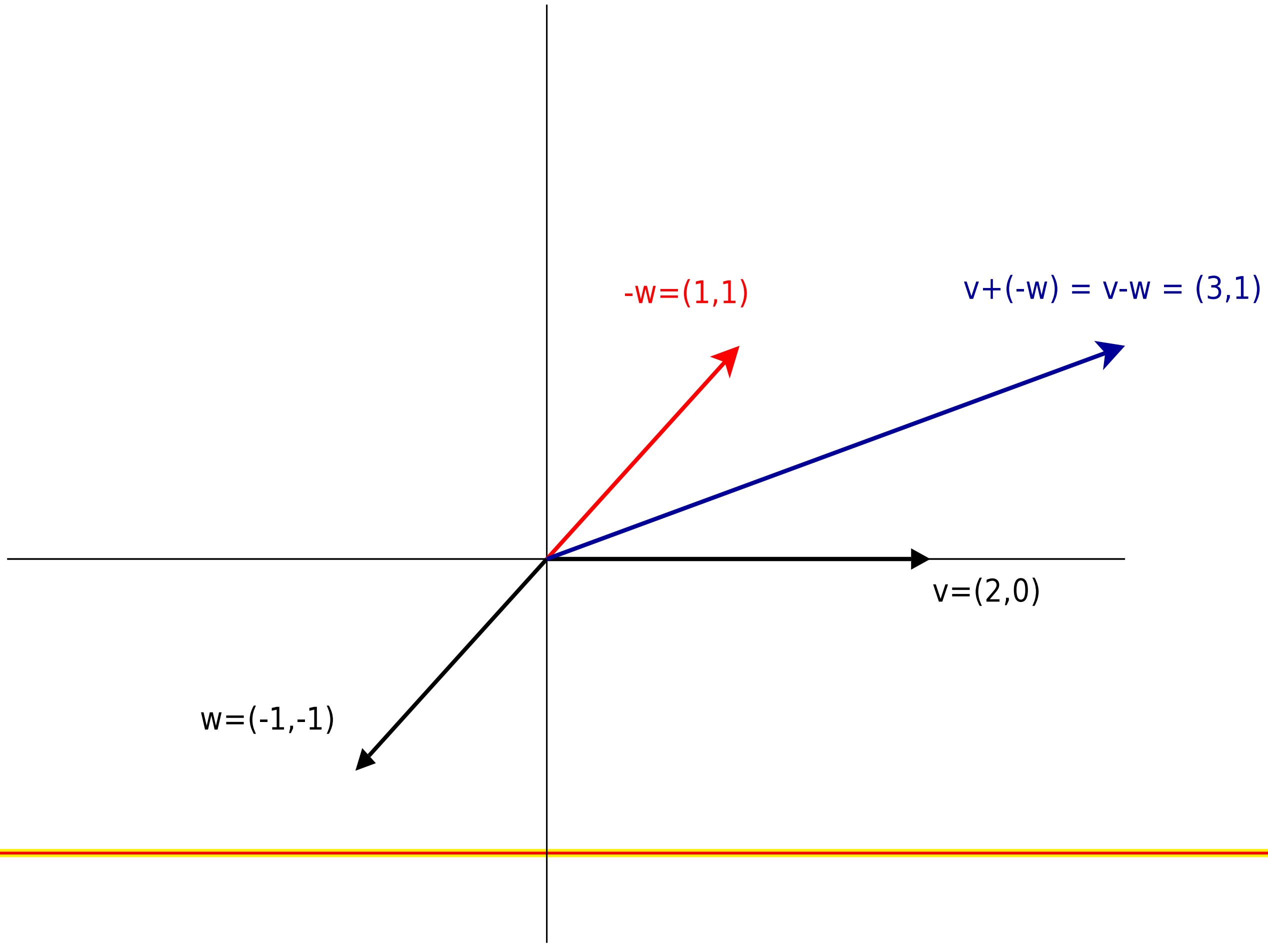
Vector Subtraction

$$\vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad -\vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



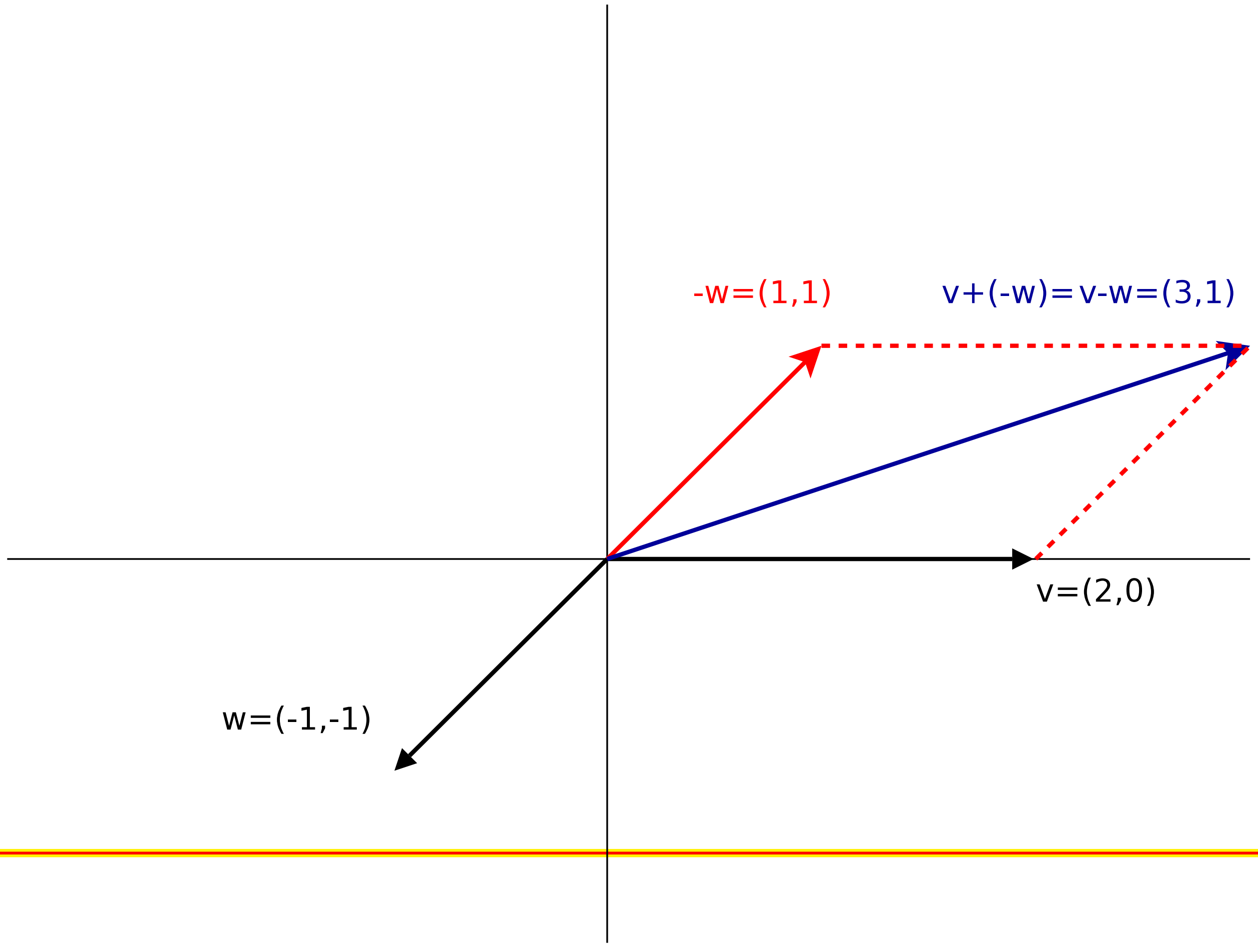
Vector Subtraction

$$\vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad \vec{v} + (-\vec{w}) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$



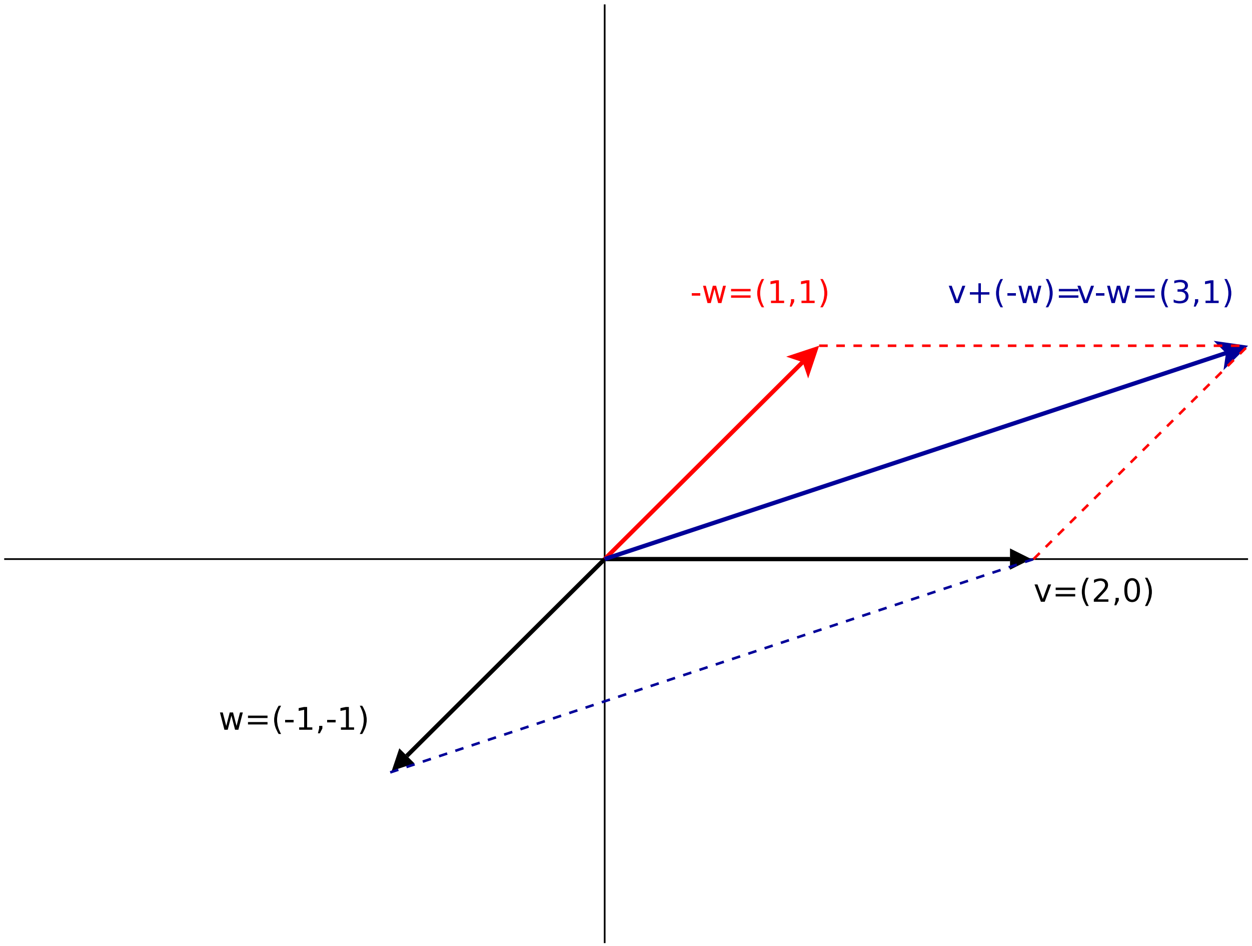
Vector Subtraction

$$\vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \quad \vec{v} - \vec{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$



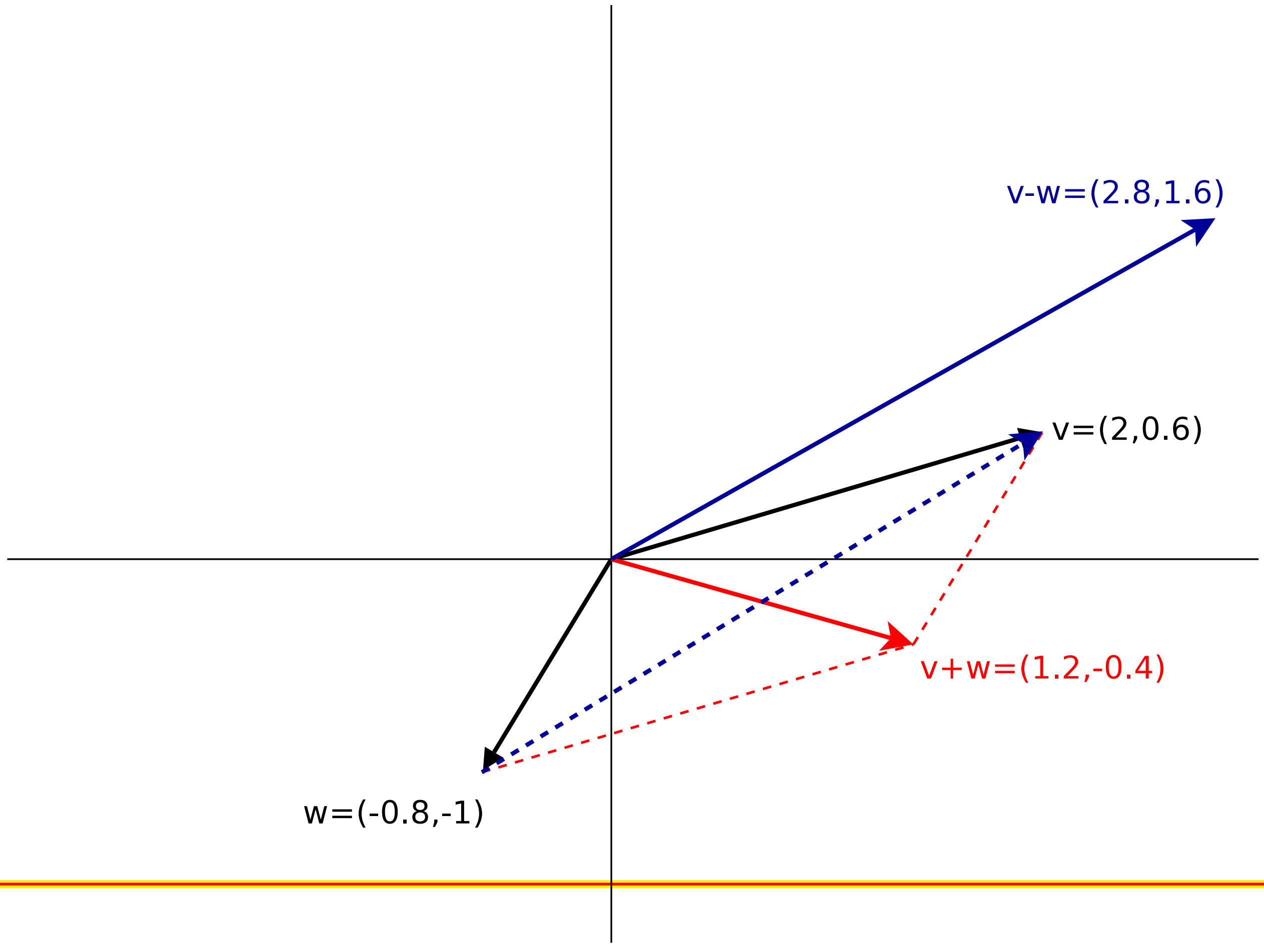
Vector Subtraction

Notice that, if we draw $\vec{v} - \vec{w}$ displaced so that it originates at the tip of $\vec{w}$, the arrowhead end terminates at the tip of $\vec{v}$.



Vector Subtraction

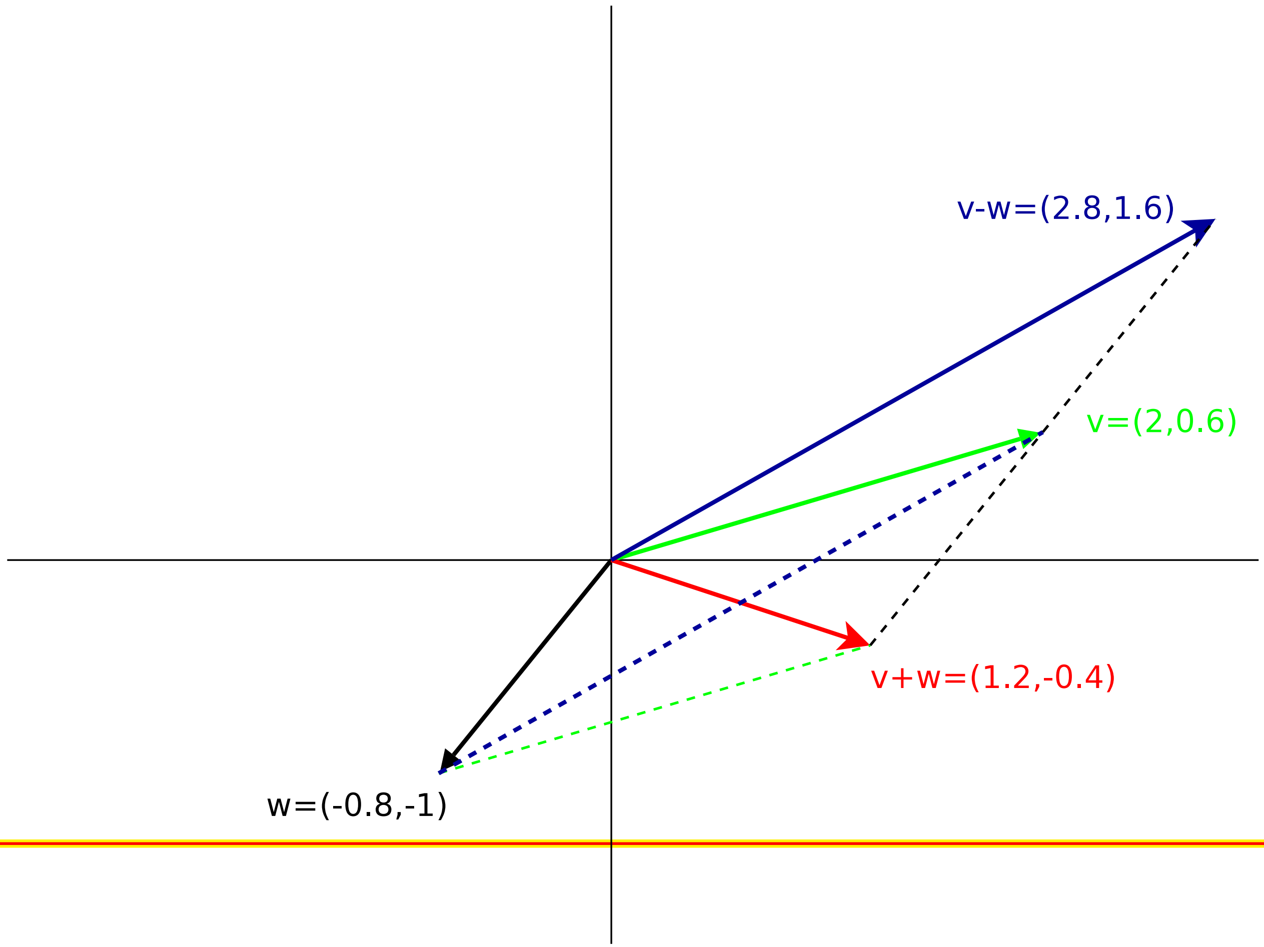
This is true in general; If we draw the parallelogram having (nonparallel) vectors $\vec{w}$ and $\vec{w}$ as sides, then $\vec{v} + \vec{w}$ always connects the origin an the opposite vertex, while $\vec{v} - \vec{w}$, **displaced to the end of** $\vec{w}$, connects the two arrow ends.



Vector Subtraction

Looking at the diagram below, can you visualize the following equation?

$$\vec{v} = \vec{w} + (\vec{v} - \vec{w})$$



Algebraic versus Geometric View

Although our definitions of vector addition and multiplication of a vector by a scalar are purely algebraic, we can visualize the results of these operations in geometric terms.

This is a common situation in Mathematics. Often there is more than one way to look at a problem.

Rather than just cluttering things up, this situation usually makes it easier to understand the Mathematics.

Some concepts are crystal clear in the geometric view, but not obvious at all in the algebraic view. For other concepts, the opposite may be true.

Algebraic Properties

The following are algebraic properties of the vector sum $\vec{v} + \vec{w}$:

For arbitrary vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$,

$$
\begin{array}{lcll}
(\vec{u} + \vec{v}) + \vec{w} & = & \vec{u} + (\vec{v} + \vec{v}) & \text{(associative)} \\
\vec{u} + \vec{w} & = & \vec{w} + \vec{u} & \text{(commutative)} \\
\vec{v} + \vec{0} & = & \vec{v} & \text{(zero element)} \\
\forall \vec{v} \in \mathbb{R}^n \exists ! \vec{x} \in \mathbb{R}^n & \text{such that} & \vec{v} + \vec{x} = \vec{0} & \text{(additive inverse)}
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\end{array}
$$

The last property is read: For every vector $\vec{v}$ in $\mathbb{R}^n$ there exists a unique vector $\vec{x}$ in $\mathbb{R}^n$ such that $\vec{v} + \vec{x}$ is the zero vector $\vec{0}$. ($\forall$ is read "for all" or "for every", $\exists$ is read "there exists a", and $!$ is read "unique")

Algebraic Properties

The following are algebraic properties of the product of a scalar and vector:

For arbitrary vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ and arbitrary scalars $c, k \in \mathbb{R}$,

$$
\begin{aligned}
k(\vec{v} + \vec{w}) &= k\vec{v} + k\vec{w} \quad \text{(distributive)} \\
(c + k)\vec{v} &= c\vec{v} + k\vec{v} \quad \text{(distributive)} \\
c(k\vec{v}) &= (ck)\vec{v} \quad \text{(associative)} \\
1\vec{v} &= \vec{v} \quad \text{multiplicative identity element}
\end{aligned}
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You should become familiar with these properties and the ones from the previous foil.