THE DETERMINANT OF AN UPPER TRIANGULAR MATRIX

Definition:

An $n \times n$ matrix A with $(ij)^{th}$ entry a_{ij} is called:

upper triangular	if	a_{ij}	$= 0$	whenever	$i > j$
lower triangular	if	a_{ij}	$= 0$	whenever	$i < j$
diagonal	if	a_{ij}	$= 0$	whenever	$i \neq j$

Theorem: The determinant of an $n \times n$ upper triangular matrix A is the product of its diagonal entries:

$$\det(A) = \prod_{i=1}^{n} a_{ii}$$

Proof. (The proof is by induction on n, the number of rows in A).

Claim 1: The determinant of a 1×1 upper triangular matrix A is

$$\prod_{i=1}^{1} a_{ii} = a_{11}$$

Proof of Claim 1: Every 1×1 matrix is upper triangular, and by definition the determinant of a 1×1 matrix A is a_{11}. (*q.e.d. claim* 1)

Claim 2: Assume that the determinant of an $n \times n$ upper triangular matrix A is:

$$\prod_{i=1}^{n} a_{ii}$$

(This statement is called the *induction hypothesis*)

Then the determinant of an $(n+1) \times (n+1)$ upper triangular matrix is

$$\prod_{i=1}^{n+1} a_{ii}$$

Proof of Claim 2: Suppose A is an $(n+1) \times (n+1)$ upper triangular matrix.

The determinant of A written as the Laplace expansion down the first column is:

$$\det(A) \;=\; \sum_{i=1}^{n+1} (-1)^{i+1} a_{i1} \det(A_{i1})$$

where A_{i1} is the matrix obtained by removing the i^{th} row and first column of A.

By hypothesis, A is upper triangular, so $a_{ij} = 0$ whenever $i > j$.

Therefore, the only nonzero element in the first column is a_{11} and the Laplace expansion reduces to:

$$\det(A) \;=\; a_{11} \det(A_{11})$$

The fact that A is $(n+1) \times (n+1)$ upper triangular means that A_{11} is $n \times n$ upper triangular.

By the induction hypothesis $\det(A_{11})$ is the product of its diagonal entries:

$$\det(A_{11}) \;=\; a_{22} \cdot a_{33} \cdots a_{(n+1)\,(n+1)} \;=\; \prod_{i=2}^{n+1} a_{ii}$$

By substitution,

$$\det(A) \;=\; a_{11} \det(A_{11}) \;=\; a_{11} \prod_{i=2}^{n+1} a_{ii} \;=\; \prod_{i=1}^{n+1} a_{ii}$$

(*q.e.d. claim* 2)

Since $P(1)$, the proposition when $n = 1$, is true, and

$$P(n) \;\Rightarrow\; P(n+1)\;,$$

the proposition is true for any positive integer n by the axiom of induction. This completes the proof of the theorem. □