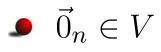


Gene Quinn

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we say that V is a (linear) **subspace** of \mathbb{R}^n if V has the following three properties:



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- V is closed under scalar multiplication

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- V is closed under scalar multiplication

If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, we have seen that the kernel of T, ker(T), is a subspace.

It is also true that the span of any nonempty set of vectors in \mathbb{R}^n is a subspace.

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The proof of these statements is as follows.

By definition, if $V = \{\vec{v}_1, \dots, \vec{v}_m\}$ is a nonempty subset of \mathbb{R}^n ,

$$span(V) = \{c_1 \vec{v}_1 + \dots + c_n \vec{v}_m : c_1, \dots, c_n \in \mathbb{R}\}$$

Claim 1: $\vec{0} \in \text{span}(V)$

Proof of Claim 1: Choose

$$c_1 = c_2 = \dots = c_n = 0$$

Then

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n \vec{v}_m = \vec{0}_n$$

so $\vec{0}_n$ is in span(V).

Claim 2: span(V) is closed under addition.

Proof of Claim 2: Let \vec{x}, \vec{y} be arbitrary vectors in span(V). Then for some scalars $c_1, \ldots, c_n, d_1, \ldots, d_n$,

 $\vec{x} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n$ and $\vec{y} = d_1 \vec{v}_1 + \dots + d_n \vec{v}_n$

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Then by the rules of vector addition and scalar multiplication,

$$\vec{x} + \vec{y} = (c_1 \vec{v}_1 + \dots + c_n \vec{v}_n) + (d_1 \vec{v}_1 + \dots + d_n \vec{v}_n)$$

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$$\vec{x} + \vec{y} = [(c_1 + d_1) \vec{v}_1 + \dots + (c_n + d_n) \vec{v}_n]$$
SO $(\vec{x} + \vec{y}) \in \text{span}(V).$

Claim 3: span(V) is closed under scalar multiplication.

Proof of Claim 2: Let \vec{x} be an arbitrary vector in span(V). Then for some scalars c_1, \ldots, c_n ,

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Now let $k \in \mathbb{R}$ be an arbitrary scalar. Then

$$k\vec{x} = k\left(c_1\vec{v}_1 + \dots + c_n\vec{v}_n\right)$$

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$$k\vec{x} = (k \cdot c_1) \, \vec{v}_1 + \dots + (k \cdot c_n) \, \vec{v}_n$$

so $k\vec{x} \in \operatorname{span}(V)$.

$\operatorname{im}(A)$ and span

If $T : \mathbb{R}^m \to \mathbb{R}^n$ with $T(\vec{x}) = A\vec{x}$, the image of this transformation, denoted by im(T) or im(A), has the property that if

 $V = \{\vec{a}_1, \ldots, \vec{a}_m\}$

is a set of vectors consisting of the m columns of the matrix $\boldsymbol{A},$ then

 $im(A) = \operatorname{span}(V)$

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(A proof of this is given in another document)

The subspaces of \mathbb{R}^2 are:

- Dimension 2: \mathbb{R}^2
- **Dimension 1:** Any line through the origin $\vec{0}_2$
- Dimension 0: $\left\{ \vec{0}_2 \right\}$ (the set consisting only of the zero vector)

The subspaces of \mathbb{R}^3 are:

- Dimension 3: \mathbb{R}^3
- Dimension 2: Any plane through the origin.
- Dimension 1: Any line through the origin.
- Dimension 0: $\left\{ \vec{0}_3 \right\}$ (the set consisting only of the zero vector)