Subspace

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Subspace

Consider a set of vectors $V \subseteq \mathbb{R}^n$

we say that V is a (linear) **subspace** of $\mathbb{R}^n$ if V has the following three properties:

- $\vec{0}_n \in V$
- V is closed under addition
- V is closed under scalar multiplication

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- V is closed under scalar multiplication

If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, we have seen that the kernel of T, $ker(T)$, is a subspace.

Subspace

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The proof of these statements is as follows.

By definition, if $V = \{\vec{v}_1, \ldots \vec{v}_m\}$ is a nonempty subset of $\mathbb{R}^n$,

$$\text{span}(V) \;=\; \{c_1\vec{v}_1 + \cdots + c_n\vec{v}_m \;:\; c_1, \ldots, c_n \in \mathbb{R}\}$$

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Claim 1: $\vec{0} \in \text{span}(V)$

Proof of Claim 1: Choose

$$c_1 = c_2 = \cdots = c_n = 0$$

Then

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_m \ = \ \vec{0}_n$$

so $\vec{0}_n$ is in span(V).

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Claim 2: span(V) is closed under addition.

Proof of Claim 2: Let $\vec{x}, \vec{y}$ be arbitrary vectors in span(V). Then for some scalars $c_1, \ldots, c_n, d_1, \ldots, d_n$,

$$\vec{x} \;=\; c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \quad \text{and} \quad \vec{y} \;=\; d_1\vec{v}_1 + \cdots + d_n\vec{v}_n$$

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Then by the rules of vector addition and scalar multiplication,

$$\vec{x} + \vec{y} = (c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) + (d_1\vec{v}_1 + \cdots + d_n\vec{v}_n)$$

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$$\vec{x} + \vec{y} \; = \; [(c_1 + d_1)\,\vec{v}_1 + \cdots + (c_n + d_n)\,\vec{v}_n]$$

so $(\vec{x} + \vec{y}) \in$ span(V).

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Claim 3: span(V) is closed under scalar multiplication.

Proof of Claim 2: Let $\vec{x}$ be an arbitrary vector in span(V). Then for some scalars $c_1, \ldots, c_n$,

$$\vec{x} \ = \ c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$$

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$$\vec{x} \;=\; c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$$

Now let $k \in \mathbb{R}$ be an arbitrary scalar. Then

$$k\vec{x} \;=\; k\left(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n\right)$$

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Now let $k \in \mathbb{R}$ be an arbitrary scalar. Then

$$k\vec{x} \ = \ k\left(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n\right)$$

or

$$k\vec{x} \ = \ (k \cdot c_1)\,\vec{v}_1 + \cdots + (k \cdot c_n)\,\vec{v}_n$$

so $k\vec{x} \in$ span(V).

$im(A)$ and span

If $T : \mathbb{R}^m \to \mathbb{R}^n$ with $T(\vec{x}) = A\vec{x}$, the image of this transformation, denoted by $im(T)$ or $im(A)$, has the property that if

$$V = \{\vec{a}_1, \ldots, \vec{a}_m\}$$

is a set of vectors consisting of the m columns of the matrix A, then

$$im(A) = \text{span}(V)$$

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(A proof of this is given in another document)

Subspaces

The subspaces of $\mathbb{R}^2$ are:

- Dimension 2: $\mathbb{R}^2$
- Dimension 1: Any line through the origin $\vec{0}_2$
- Dimension 0: $\left\{\vec{0}_2\right\}$ (the set consisting only of the zero vector)

Subspaces

The subspaces of $\mathbb{R}^3$ are:

- Dimension 3: $\mathbb{R}^3$
- Dimension 2: Any plane through the origin.
- Dimension 1: Any line through the origin.
- Dimension 0: $\left\{\vec{0}_3\right\}$ (the set consisting only of the zero vector)