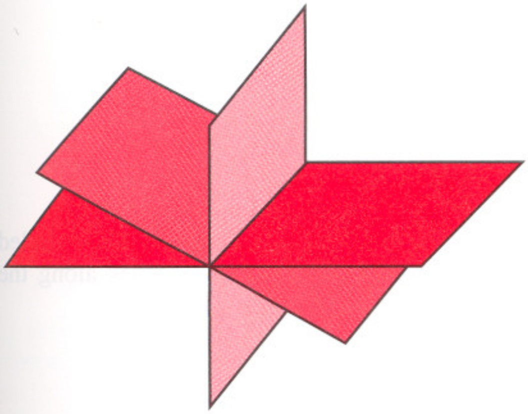
CHAPTER 1

Linear Equations



1.1 INTRODUCTION TO LINEAR SYSTEMS

Traditionally, algebra was the art of solving equations and systems of equations. The word *algebra* comes from the Arabic *al-jabr,* which means *restoration* (of broken parts).[1] The term was first used in a mathematical sense by Mohammed al-Khowarizmi (c. 780–850), who worked at the House of Wisdom, an academy established by Caliph al-Ma'mun in Baghdad. Linear algebra, then, is the art of solving systems of linear equations.

The need to solve systems of linear equations frequently arises in mathematics, statistics, physics, astronomy, engineering, computer science, and economics.

Solving systems of linear equations is not conceptually difficult. For small systems, ad hoc methods certainly suffice. Larger systems, however, require more systematic methods. The approach generally used today was beautifully explained 2,000 years ago in a Chinese text, the *Nine Chapters on the Mathematical Art* (Jiuzhang Suanshu, 九章算術).[2] Chapter 8 of that text, called *Method of Rectangular Arrays* (Fang Cheng, 方程), contains the following problem:

> The yield of one bundle of inferior rice, two bundles of medium grade rice, and three bundles of superior rice is 39 *dou* of grain.[3] The yield of one bundle of inferior rice, three bundles of medium grade rice, and two bundles of superior rice is 34 *dou*. The yield of three bundles of inferior rice, two bundles of medium grade rice, and one bundle of superior rice is 26 *dou*. What is the yield of one bundle of each grade of rice?

In this problem the unknown quantities are the yields of one bundle of inferior, one bundle of medium grade, and one bundle of superior rice. Let us denote these quantities by x, y, and z, respectively. The problem can then be represented by the

[1] At one time, it was not unusual to see the sign *Algebrista y Sangrador* (bone setter and blood letter) at the entrance of a Spanish barber's shop.

[2] Shen Kangshen et al. (ed.), *The Nine Chapters on the Mathematical Art*, Companion and Commentary, Oxford University Press, 1999.

[3] The *dou* is a measure of volume, corresponding to about 2 liters at that time.

following system of linear equations:

$$\begin{vmatrix} x + 2y + 3z = 39 \\ x + 3y + 2z = 34 \\ 3x + 2y + z = 26 \end{vmatrix}.$$

To solve for x, y, and z, we need to transform this system from the form

$$\begin{vmatrix} x + 2y + 3z = 39 \\ x + 3y + 2z = 34 \\ 3x + 2y + z = 26 \end{vmatrix} \quad \text{into the form} \quad \begin{vmatrix} x \quad\quad = \ldots \\ \quad y \quad = \ldots \\ \quad\quad z = \ldots \end{vmatrix}.$$

In other words, we need to eliminate the terms that are off the diagonal, those circled in the following equations, and make the coefficients of the variables along the diagonal equal to 1:

$$\begin{aligned} x &+ \textcircled{2y} + \textcircled{3z} = 39 \\ \textcircled{x} &+ 3y + \textcircled{2z} = 34 \\ \textcircled{3x} &+ \textcircled{2y} + z = 26. \end{aligned}$$

We can accomplish these goals step by step, one variable at a time. In the past, you may have simplified systems of equations by adding equations to one another or subtracting them. In this system, we can eliminate the variable x from the second equation by subtracting the first equation from the second:

$$\begin{vmatrix} x + 2y + 3z = 39 \\ x + 3y + 2z = 34 \\ 3x + 2y + z = 26 \end{vmatrix} \quad \xrightarrow[-1\text{st equation}]{} \quad \begin{vmatrix} x + 2y + 3z = 39 \\ y - z = -5 \\ 3x + 2y + z = 26 \end{vmatrix}.$$

To eliminate the variable x from the third equation, we subtract the first equation from the third equation three times. We multiply the first equation by 3 to get

$$3x + 6y + 9z = 117 \qquad (3 \times 1\text{st equation})$$

and then subtract this result from the third equation:

$$\begin{vmatrix} x + 2y + 3z = 39 \\ y - z = -5 \\ 3x + 2y + z = 26 \end{vmatrix} \quad \xrightarrow[-3 \times 1\text{st equation}]{} \quad \begin{vmatrix} x + 2y + 3z = 39 \\ y - z = -5 \\ -4y - 8z = -91 \end{vmatrix}.$$

Similarly, we eliminate the variable y above and below the diagonal:

$$\begin{vmatrix} x + 2y + 3z = 39 \\ y - z = -5 \\ -4y - 8z = -91 \end{vmatrix} \quad \xrightarrow[+4 \times 2\text{nd equation}]{-2 \times 2\text{nd equation}} \quad \begin{vmatrix} x + 5z = 49 \\ y - z = -5 \\ -12z = -111 \end{vmatrix}.$$

Before we eliminate the variable z above the diagonal, we make the coefficient of z on the diagonal equal to 1, by dividing the last equation by -12:

$$\begin{vmatrix} x + 5z = 49 \\ y - z = -5 \\ -12z = -111 \end{vmatrix} \quad \xrightarrow[\div(-12)]{} \quad \begin{vmatrix} x + 5z = 49 \\ y - z = -5 \\ z = 9.25 \end{vmatrix}.$$

Finally, we eliminate the variable z above the diagonal:

$$\begin{vmatrix} x + 5z = 49 \\ y - z = -5 \\ z = 9.25 \end{vmatrix} \quad \xrightarrow{\substack{-5 \times \text{third equation} \\ + \text{third equation}}} \quad \begin{vmatrix} x = 2.75 \\ y = 4.25 \\ z = 9.25 \end{vmatrix}.$$

The yields of inferior, medium grade, and superior rice are 2.75, 4.25, and 9.25 *dou* per bundle, respectively.

By substituting these values, we can check that $x = 2.75$, $y = 4.25$, $z = 9.25$ is indeed the solution of the system:

$$\begin{aligned} 2.75 + 2 \times 4.25 + 3 \times 9.25 &= 39 \\ 2.75 + 3 \times 4.25 + 2 \times 9.25 &= 34 \\ 3 \times 2.75 + 2 \times 4.25 + \phantom{2 \times{}} 9.25 &= 26. \end{aligned}$$

Happily, in linear algebra, you are almost always able to check your solutions. It will help you if you get into the habit of checking now.

Geometric Interpretation

How can we interpret this result geometrically? Each of the three equations of the system defines a plane in x–y–z-space. The solution set of the system consists of those points (x, y, z) that lie in all three planes (i.e., the intersection of the three planes). Algebraically speaking, the solution set consists of those ordered triples of numbers (x, y, z) that satisfy all three equations simultaneously. Our computations show that the system has only one solution, $(x, y, z) = (2.75, 4.25, 9.25)$. This means that the planes defined by the three equations intersect at the point $(x, y, z) = (2.75, 4.25, 9.25)$, as shown in Figure 1.

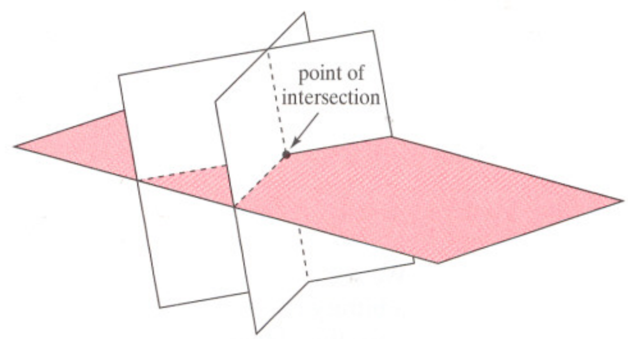


Figure 1 Three planes in space, intersecting at a point.

While three different planes in space usually intersect at a point, they may have a line in common (see Figure 2a) or may not have a common intersection at all, as shown in Figure 2b. Therefore, a system of three equations with three unknowns may have a unique solution, infinitely many solutions, or no solutions at all.

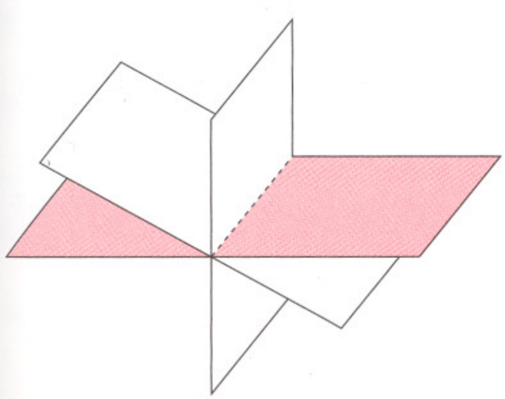


Figure 2(a) Three planes having a line in common.

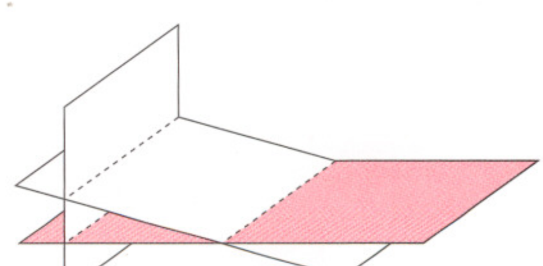


Figure 2(b) Three planes with no common intersection.

A System with Infinitely Many Solutions

Next, let's consider a system of linear equations that has infinitely many solutions:

$$\left|\begin{aligned} 2x + 4y + 6z &= 0 \\ 4x + 5y + 6z &= 3 \\ 7x + 8y + 9z &= 6 \end{aligned}\right|.$$

We can solve this system using elimination as previously discussed. For simplicity, we label the equations with Roman numerals.

$$\left|\begin{aligned} 2x + 4y + 6z &= 0 \\ 4x + 5y + 6z &= 3 \\ 7x + 8y + 9z &= 6 \end{aligned}\right| \begin{array}{c} \div 2 \\ \longrightarrow \\ \ \end{array} \left|\begin{aligned} x + 2y + 3z &= 0 \\ 4x + 5y + 6z &= 3 \\ 7x + 8y + 9z &= 6 \end{aligned}\right| \begin{array}{c} \longrightarrow \\ -4\,(\mathrm{I}) \\ -7\,(\mathrm{I}) \end{array}$$

$$\left|\begin{aligned} x + 2y + 3z &= 0 \\ -3y - 6z &= 3 \\ -6y - 12z &= 6 \end{aligned}\right| \begin{array}{c} \longrightarrow \\ \div(-3) \\ \ \end{array} \left|\begin{aligned} x + 2y + 3z &= 0 \\ y + 2z &= -1 \\ -6y - 12z &= 6 \end{aligned}\right| \begin{array}{c} -2\,(\mathrm{II}) \\ \longrightarrow \\ +6\,(\mathrm{II}) \end{array}$$

$$\left|\begin{aligned} x \quad - z &= 2 \\ y + 2z &= -1 \\ 0 &= 0 \end{aligned}\right| \longrightarrow \left|\begin{aligned} x \quad - z &= 2 \\ y + 2z &= -1 \end{aligned}\right|$$

After omitting the trivial equation $0 = 0$, we are left with only two equations with three unknowns. The solution set is the intersection of two nonparallel planes in space (i.e., a line). This system has infinitely many solutions.

The two foregoing equations can be written as follows:

$$\left|\begin{aligned} x &= z + 2 \\ y &= -2z - 1 \end{aligned}\right|.$$

We see that both x and y are determined by z. We can freely choose a value of z, an arbitrary real number; then the two preceding equations give us the values of x and y for this choice of z. For example,

- Choose $z = 1$. Then $x = z + 2 = 3$ and $y = -2z - 1 = -3$. The solution is $(x, y, z) = (3, -3, 1)$.
- Choose $z = 7$. Then $x = z + 2 = 9$ and $y = -2z - 1 = -15$. The solution is $(x, y, z) = (9, -15, 7)$.

More generally, if we choose $z = t$, an arbitrary real number, we get $x = t + 2$ and $y = -2t - 1$. Therefore, the general solution is

$$(x, y, z) = (t + 2, -2t - 1, t) = (2, -1, 0) + t(1, -2, 1).$$

This equation represents a line in space, as shown in Figure 3.

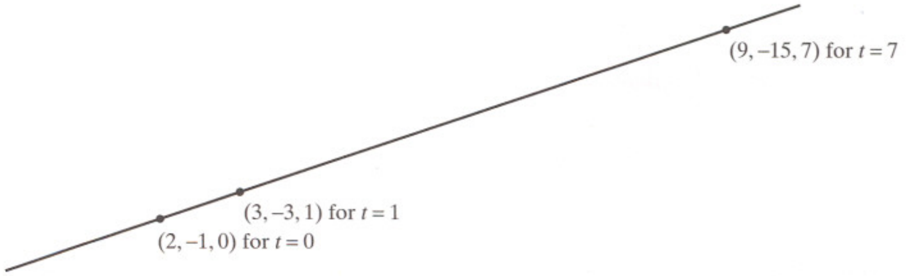


Figure 3 The line $(x, y, z) = (t + 2, -2t - 1, t)$.

A System without Solutions

In the following system, perform the eliminations yourself to obtain the result shown:

$$\left| \begin{array}{l} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 3 \\ 7x + 8y + 9z = 0 \end{array} \right| \quad \longrightarrow \quad \left| \begin{array}{rcr} x \qquad - z &=& 2 \\ y + 2z &=& -1 \\ 0 &=& -6 \end{array} \right| .$$

Whatever values we choose for x, y, and z, the equation $0 = -6$ cannot be satisfied. This system is *inconsistent*; that is, it has no solutions.

EXERCISES 1.1

GOAL Set up and solve systems with as many as three linear equations with three unknowns, and interpret the equations and their solutions geometrically.

In Exercises 1 through 10, find all solutions of the linear systems using elimination as discussed in this section. Then check your solutions.

1. $\left| \begin{array}{l} x + 2y = 1 \\ 2x + 3y = 1 \end{array} \right|$

2. $\left| \begin{array}{l} 4x + 3y = 2 \\ 7x + 5y = 3 \end{array} \right|$

3. $\left| \begin{array}{l} 2x + 4y = 3 \\ 3x + 6y = 2 \end{array} \right|$

4. $\left| \begin{array}{l} 2x + 4y = 2 \\ 3x + 6y = 3 \end{array} \right|$

5. $\left| \begin{array}{l} 2x + 3y = 0 \\ 4x + 5y = 0 \end{array} \right|$

6. $\left| \begin{array}{l} x + 2y + 3z = 8 \\ x + 3y + 3z = 10 \\ x + 2y + 4z = 9 \end{array} \right|$

7. $\left| \begin{array}{l} x + 2y + 3z = 1 \\ x + 3y + 4z = 3 \\ x + 4y + 5z = 4 \end{array} \right|$

8. $\left| \begin{array}{l} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \\ 7x + 8y + 10z = 0 \end{array} \right|$

9. $\left| \begin{array}{l} x + 2y + 3z = 1 \\ 3x + 2y + z = 1 \\ 7x + 2y - 3z = 1 \end{array} \right|$

10. $\left| \begin{array}{l} x + 2y + 3z = 1 \\ 2x + 4y + 7z = 2 \\ 3x + 7y + 11z = 8 \end{array} \right|$

In Exercises 11 through 13, find all solutions of the linear systems. Represent your solutions graphically, as intersections of lines in the x–y-plane.

11. $\left| \begin{array}{l} x - 2y = 2 \\ 3x + 5y = 17 \end{array} \right|$

12. $\left| \begin{array}{l} x - 2y = 3 \\ 2x - 4y = 6 \end{array} \right|$

13. $\left| \begin{array}{l} x - 2y = 3 \\ 2x - 4y = 8 \end{array} \right|$

In Exercises 14 through 16, find all solutions of the linear systems. Describe your solutions in terms of intersecting planes. You need not sketch these planes.

14. $\left| \begin{array}{l} x + 4y + z = 0 \\ 4x + 13y + 7z = 0 \\ 7x + 22y + 13z = 1 \end{array} \right|$

15. $\left| \begin{array}{l} x + y - z = 0 \\ 4x - y + 5z = 0 \\ 6x + y + 4z = 0 \end{array} \right|$

16. $\left| \begin{array}{l} x + 4y + z = 0 \\ 4x + 13y + 7z = 0 \\ 7x + 22y + 13z = 0 \end{array} \right|$

17. Find all solutions of the linear system

$$\left| \begin{array}{l} x + 2y = a \\ 3x + 5y = b \end{array} \right| ,$$

where a and b are arbitrary constants.

18. Find all solutions of the linear system

$$\left| \begin{array}{l} x + 2y + 3z = a \\ x + 3y + 8z = b \\ x + 2y + 2z = c \end{array} \right| ,$$

where a, b, and c are arbitrary constants.

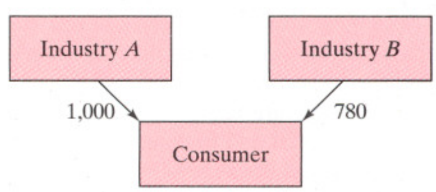
19. Consider a two-commodity market. When the unit prices of the products are P_1 and P_2, the quantities demanded, D_1 and D_2, and the quantities supplied, S_1 and S_2, are given by

$$\begin{array}{ll} D_1 = 70 - 2P_1 + P_2, & S_1 = -14 + 3P_1, \\ D_2 = 105 + P_1 - P_2, & S_2 = -7 \qquad\qquad + 2P_2. \end{array}$$

a. What is the relationship between the two commodities? Do they compete, as do Volvos and BMWs, or do they complement one another, as do shirts and ties?

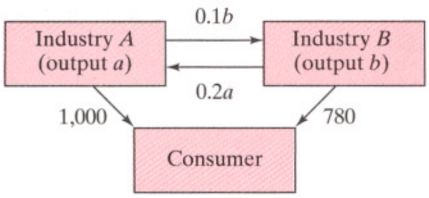
b. Find the equilibrium prices (i.e., the prices for which supply equals demand), for both products.

20. The Russian-born U.S. economist and Nobel laureate Wassily Leontief (1906–1999) was interested in the following question: What output should each of the industries in an economy produce to satisfy the total demand for all products? Here, we consider a very simple example of input-output analysis, an economy with only two industries, A and B. Assume that the consumer demand for their products is, respectively, 1,000 and 780, in millions of dollars per year.

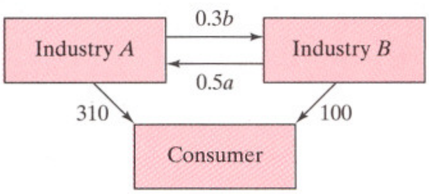


What outputs a and b (in millions of dollars per year) should the two industries generate to satisfy the demand?

You may be tempted to say 1,000 and 780, respectively, but things are not quite as simple as that. We have to take into account the interindustry demand as well. Let us say that industry A produces electricity. Of course, producing almost any product will require electric power. Suppose that industry B needs 10¢ worth of electricity for each \$1 of output B produces and that industry A needs 20¢ worth of B's products for each \$1 of output A produces. Find the outputs a and b needed to satisfy both consumer and interindustry demand.



21. Find the outputs a and b needed to satisfy the consumer and interindustry demands given in the following figure (see Exercise 20):



22. Consider the differential equation

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - x = \cos(t).$$

This equation could describe a forced damped oscillator, as we will see in Chapter 9. We are told that the differential equation has a solution of the form

$$x(t) = a\sin(t) + b\cos(t).$$

Find a and b, and graph the solution.

23. Find all solutions of the system

$$\begin{vmatrix} 7x - y = \lambda x \\ -6x + 8y = \lambda y \end{vmatrix}, \quad \text{for}$$

a. $\lambda = 5$ b. $\lambda = 10$, and c. $\lambda = 15$.

24. On your next trip to Switzerland, you should take the scenic boat ride from Rheinfall to Rheinau and back. The trip downstream from Rheinfall to Rheinau takes 20 minutes, and the return trip takes 40 minutes; the distance between Rheinfall and Rheinau along the river is 8 kilometers. How fast does the boat travel (relative to the water), and how fast does the river Rhein flow in this area? You may assume both speeds to be constant throughout the journey.

25. Consider the linear system

$$\begin{vmatrix} x + y - z = -2 \\ 3x - 5y + 13z = 18 \\ x - 2y + 5z = k \end{vmatrix},$$

where k is an arbitrary number.

a. For which value(s) of k does this system have one or infinitely many solutions?

b. For each value of k you found in part a, how many solutions does the system have?

c. Find all solutions for each value of k.

26. Consider the linear system

$$\begin{vmatrix} x + y - z = 2 \\ x + 2y + z = 3 \\ x + y + (k^2 - 5)z = k \end{vmatrix},$$

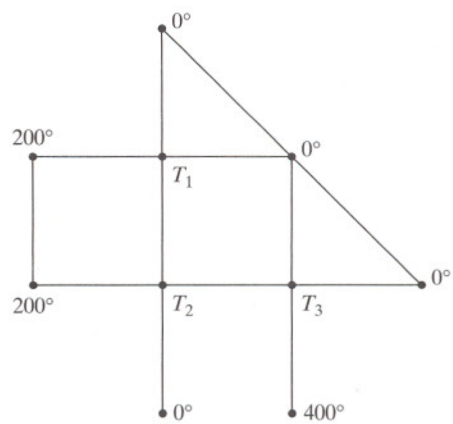
where k is an arbitrary constant. For which value(s) of k does this system have a unique solution? For which value(s) of k does the system have infinitely many solutions? For which value(s) of k is the system inconsistent?

27. Emile and Gertrude are brother and sister. Emile has twice as many sisters as brothers, and Gertrude has just as many brothers as sisters. How many children are there in this family?

28. In a grid of wires, the temperature at exterior mesh points is maintained at constant values (in °C) as shown in the accompanying figure. When the grid is in thermal equilibrium, the temperature T at each interior mesh point is the average of the temperatures at the four adjacent points. For example,

$$T_2 = \frac{T_3 + T_1 + 200 + 0}{4}.$$

Find the temperatures T_1, T_2, and T_3 when the grid is in thermal equilibrium.



29. Find the polynomial of degree 2 (a polynomial of the form $f(t) = a + bt + ct^2$) whose graph goes through the points $(1, -1)$, $(2, 3)$, and $(3, 13)$. Sketch the graph of this polynomial.

30. Find a polynomial of degree ≤ 2 [a polynomial of the form $f(t) = a + bt + ct^2$] whose graph goes through the points $(1, p)$, $(2, q)$, $(3, r)$, where p, q, r are arbitrary constants. Does such a polynomial exist for all values of p, q, r?

31. Find all the polynomials $f(t)$ of degree ≤ 2 whose graphs run through the points $(1, 3)$ and $(2, 6)$, such that $f'(1) = 1$ [where $f'(t)$ denotes the derivative].

32. Find all the polynomials $f(t)$ of degree ≤ 2 whose graphs run through the points $(1, 1)$ and $(2, 0)$, such that $\int_1^2 f(t)\,dt = -1$.

33. Find all the polynomials $f(t)$ of degree ≤ 2 whose graphs run through the points $(1, 1)$ and $(3, 3)$, such that $f'(2) = 1$.

34. Find all the polynomials $f(t)$ of degree ≤ 2 whose graphs run through the points $(1, 1)$ and $(3, 3)$, such that $f'(2) = 3$.

35. Find the function $f(t)$ of the form $f(t) = ae^{3t} + be^{2t}$ such that $f(0) = 1$ and $f'(0) = 4$.

36. Find the function $f(t)$ of the form $f(t) = a\cos(2t) + b\sin(2t)$ such that $f''(t) + 2f'(t) + 3f(t) = 17\cos(2t)$. (This is the kind of differential equation you might have to solve when dealing with forced damped oscillators, in physics or engineering.)

37. Find all points (a, b, c) in space for which the system

$$\left| \begin{array}{l} x + 2y + 3z = a \\ 4x + 5y + 6z = b \\ 7x + 8y + 9z = c \end{array} \right|$$

has at least one solution.

38. Linear systems are particularly easy to solve when they are in *triangular* form (i.e., all entries above or below the diagonal are zero).

a. Solve the lower triangular system

$$\left| \begin{array}{rcl} x_1 & = & -3 \\ -3x_1 + x_2 & = & 14 \\ x_1 + 2x_2 + x_3 & = & 9 \\ -x_1 + 8x_2 - 5x_3 + x_4 & = & 33 \end{array} \right|$$

by forward substitution, finding x_1 first, then x_2, then x_3, and finally x_4.

b. Solve the upper triangular system

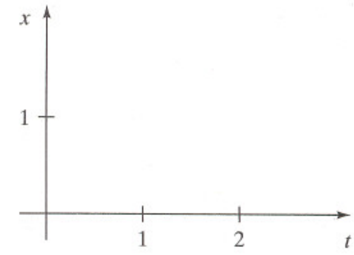
$$\left| \begin{array}{rcl} x_1 + 2x_2 - x_3 + 4x_4 & = & -3 \\ x_2 + 3x_3 + 7x_4 & = & 5 \\ x_3 + 2x_4 & = & 2 \\ x_4 & = & 0 \end{array} \right| .$$

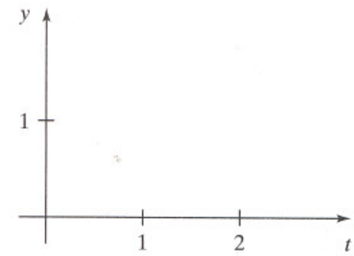
39. Consider the linear system

$$\left| \begin{array}{l} x + y = 1 \\ x + \frac{t}{2}y = t \end{array} \right| ,$$

where t is a nonzero constant.

a. Determine the x- and y-intercepts of the lines $x + y = 1$ and $x + (t/2)y = t$; sketch these lines. For which values of the constant t do these lines intersect? For these values of t, the point of intersection (x, y) depends on the choice of the constant t; that is, we can consider x and y as functions of t. Draw rough sketches of these functions.





Explain briefly how you found these graphs. Argue geometrically, without solving the system algebraically.

b. Now solve the system algebraically. Verify that the graphs you sketched in part (a) are compatible with your algebraic solution.

40. Find a system of linear equations with three unknowns whose solutions are the points on the line through $(1, 1, 1)$ and $(3, 5, 0)$.

41. Find a system of linear equations with three unknowns x, y, z whose solutions are

$$x = 6 + 5t, \quad y = 4 + 3t, \quad \text{and} \quad z = 2 + t,$$

where t is an arbitrary constant.

42. Boris and Marina are shopping for chocolate bars. Boris observes, "If I add half my money to yours, it will be enough to buy two chocolate bars." Marina naively asks, "If I add half my money to yours, how many can we buy?" Boris replies, "One chocolate bar." How much money did Boris have? (From Yuri Chernyak and Robert Rose: *The Chicken from Minsk*, Basic Books, New York, 1995.)

43. Here is another method to solve a system of linear equations: Solve one of the equations for one of the variables, and substitute the result into the other equations. Repeat this process until you run out of variables or equations. Consider the example discussed on page 2:

$$\left| \begin{array}{l} x + 2y + 3z = 39 \\ x + 3y + 2z = 34 \\ 3x + 2y + z = 26 \end{array} \right| .$$

We can solve the first equation for x:

$$x = 39 - 2y - 3z.$$

Then we substitute this equation into the other equations:

$$\begin{vmatrix} (39 - 2y - 3z) + 3y + 2z = 34 \\ 3(39 - 2y - 3z) + 2y + z = 26 \end{vmatrix}.$$

We can simplify:

$$\begin{vmatrix} y - z = -5 \\ -4y - 8z = -91 \end{vmatrix}.$$

Now, $y = z - 5$, so that $-4(z - 5) - 8z = -91$, or

$$-12z = -111.$$

We find that $z = \frac{111}{12} = 9.25$. Then

$$y = z - 5 = 4.25,$$

and

$$x = 39 - 2y - 3z = 2.75.$$

Explain why this method is essentially the same as the method discussed in this section, only the bookkeeping is different.

44. A hermit eats only two kinds of food: brown rice and yogurt. The rice contains 3 grams of protein and 30 grams of carbohydrates per serving, while the yogurt contains 12 grams of protein and 20 grams of carbohydrates.

a. If the hermit wants to take in 60 grams of protein and 300 grams of carbohydrates per day, how many servings of each item should he consume?

b. If the hermit wants to take in P grams of protein and C grams of carbohydrates per day, how many servings of each item should he consume?

45. I have 32 bills in my wallet, in the denominations of US$ 1, 5, and 10, worth \$100 in total. How many do I have of each denomination?

46. Some parking meters in Milan, Italy, accept coins in the denominations of 20 cent, 50 cent, and 2 Euro. As an incentive program, the city administrators offer a big reward (a brand new red Ferrari) to any meter maid who brings back exactly 1000 coins worth exactly 1000 Euro from the daily rounds. What are the odds of this reward being claimed anytime soon?

1.2 MATRICES, VECTORS, AND GAUSS–JORDAN ELIMINATION

When mathematicians in ancient China had to solve a system of simultaneous linear equations such as[4]

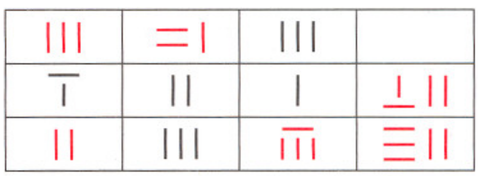
$$\begin{vmatrix} 3x + 21y - 3z = 0 \\ -6x - 2y - z = 62 \\ 2x - 3y + 8z = 32 \end{vmatrix},$$

they took all the numbers involved in this system and arranged them in a rectangular pattern (*Fang Cheng* in Chinese), as follows[5]:

3	21	−3	0
−6	−2	−1	62
2	−3	8	32

All the information about this system is conveniently stored in this array of numbers.

The entries were represented by counting rods, as shown below; red and black rods stand for positive and negative numbers, respectively. (Can you detect how this number system works?) The equations were then solved in a hands-on fashion, by manipulating the rods. We leave it to the reader to find the solution.



[4]This example is taken from Chapter 8 of the *Nine Chapters on the Mathematical Art*; see page 1. Our source is George Gheverghese Joseph, *The Crest of the Peacock; Non-European Roots of Mathematics*, 2nd ed., Princeton University Press, 2000.

[5]Actually, the roles of rows and columns were reversed in the Chinese representation.

Today, such a rectangular array of numbers,

$$\begin{bmatrix} 3 & 21 & -3 & 0 \\ -6 & -2 & -1 & 62 \\ 2 & -3 & 8 & 32 \end{bmatrix},$$

is called a **matrix**.[6] Since this particular matrix has three rows and four columns, it is called a 3×4 matrix ("three by four").

The four columns of the matrix

The three rows of the matrix

$$\begin{bmatrix} 3 & 21 & -3 & 0 \\ -6 & -2 & -1 & 62 \\ 2 & -3 & 8 & 32 \end{bmatrix}$$

Note that the first column of this matrix corresponds to the first variable of the system, while the first row corresponds to the first equation.

It is customary to label the entries of a 3×4 matrix A with double subscripts as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}.$$

The first subscript refers to the row, and the second to the column: The entry a_{ij} is located in the ith row and the jth column.

Two matrices A and B are equal if they are the same size and if corresponding entries are equal: $a_{ij} = b_{ij}$.

If the number of rows of a matrix A equals the number of columns (A is $n \times n$), then A is called a *square matrix,* and the entries $a_{11}, a_{22}, \ldots, a_{nn}$ form the (main) *diagonal* of A. A square matrix A is called *diagonal* if all its entries above and below the main diagonal are zero; that is, $a_{ij} = 0$ whenever $i \neq j$. A square matrix A is called *upper triangular* if all its entries below the main diagonal are zero; that is, $a_{ij} = 0$ whenever i exceeds j. *Lower triangular* matrices are defined analogously. A matrix whose entries are all zero is called a *zero matrix* and is denoted by 0 (regardless of its size). Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 5 & 0 & 0 \\ 4 & 0 & 0 \\ 3 & 2 & 1 \end{bmatrix}.$$

The matrices B, C, D, and E are square, C is diagonal, C and D are upper triangular, and C and E are lower triangular.

Matrices with only one column or row are of particular interest.

[6]It appears that the term *matrix* was first used in this sense by the English mathematician J. J. Sylvester, in 1850.

Vectors

A matrix with only one column is called a column vector, or simply a *vector*. The entries of a vector are called its *components*. The set of all column vectors with n components is denoted by $\mathbb{R}^n$.

A matrix with only one row is called a *row vector*.

In this text, the term *vector* refers to column vectors, unless otherwise stated. The reason for our preference for column vectors will become apparent in the next section.

Examples of vectors are

$$\begin{bmatrix} 1 \\ 2 \\ 9 \\ 1 \end{bmatrix},$$

a (column) vector in $\mathbb{R}^4$, and

$$\begin{bmatrix} 1 & 5 & 5 & 3 & 7 \end{bmatrix},$$

a row vector with five components. Note that the m columns of an $n \times m$ matrix are vectors in $\mathbb{R}^n$.

In previous courses in mathematics or physics, you may have thought about vectors from a more geometric point of view. (See the Appendix for a summary of basic facts on vectors.) Let's establish some conventions regarding the geometric representation of vectors.

The *standard representation* of a vector

$$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix},$$

in the Cartesian coordinate plane is as an *arrow* (a directed line segment) connecting the origin to the point (x, y), as shown in Figure 1.

Occasionally, it is helpful to translate (or shift) the vector in the plane (preserving its direction and length), so that it will connect some point (a, b) to the point $(a + x, b + y)$, as shown in Figure 2.

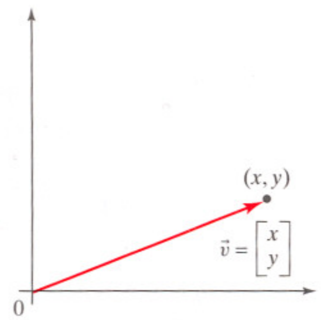


Figure 1

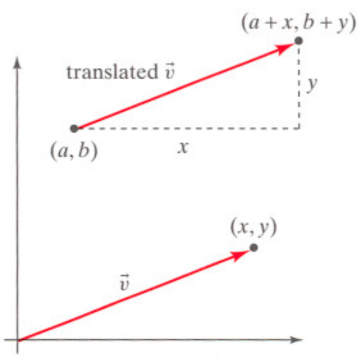


Figure 2

In this text, we consider the *standard representation* of vectors, unless we explicitly state that the vector has been translated.

When considering an infinite set of vectors, the arrow representation becomes impractical. In this case, it is sensible to represent the vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ simply by the point (x, y), the head of the standard arrow representation of $\vec{v}$.

For example, the set of all vectors $\vec{v} = \begin{bmatrix} x \\ x+1 \end{bmatrix}$ (where x is arbitrary) can be represented as the *line* $y = x + 1$. For a few special values of x we may still use the arrow representation, as illustrated in Figure 3.

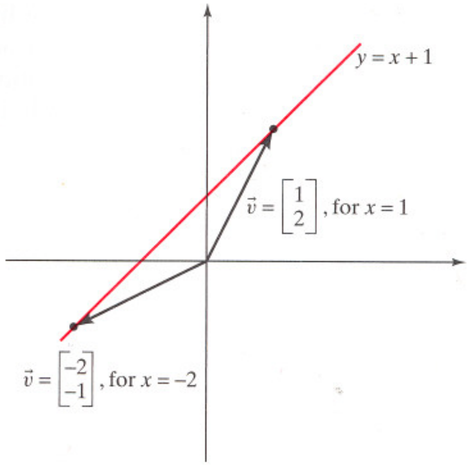


Figure 3

In this course, it will often be helpful to think about a vector numerically, as a list of numbers, which we will usually write in a column.

In our digital age, information is often transmitted and stored as a string of numbers (i.e., as a vector). A section of ten seconds of music on a CD is stored as a vector with 440,000 components. A weather photograph taken by a satellite is transmitted to Earth as a string of numbers.

Consider the system

$$\begin{vmatrix} 2x + 8y + 4z = 2 \\ 2x + 5y + z = 5 \\ 4x + 10y - z = 1 \end{vmatrix}.$$

Sometimes we are interested in the matrix

$$\begin{bmatrix} 2 & 8 & 4 \\ 2 & 5 & 1 \\ 4 & 10 & -1 \end{bmatrix},$$

which contains the coefficients of the system, called its *coefficient matrix.*

By contrast, the matrix

$$\begin{bmatrix} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{bmatrix},$$

which displays all the numerical information contained in the system, is called its *augmented matrix*. For the sake of clarity, we will often indicate the position of the equal signs in the equations by a dotted line:

$$\left[\begin{array}{ccc|c} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array}\right].$$

To solve the system, it is more efficient to perform the elimination on the augmented matrix rather than on the equations themselves. Conceptually, the two approaches are equivalent, but working with the augmented matrix requires less writing yet is easier to read, with some practice. Instead of dividing an *equation* by a scalar,[7] you can divide a *row* by a scalar. Instead of adding a multiple of an equation to another equation, you can add a multiple of a row to another row.

As you perform elimination on the augmented matrix, you should always remember the linear system lurking behind the matrix. To illustrate this method, we perform the elimination both on the augmented matrix and on the linear system it represents:

$$\left[\begin{array}{ccc|c} 2 & 8 & -4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array}\right] \begin{array}{l} \div 2 \\ \\ \\ \end{array} \qquad \left|\begin{array}{rrrr} 2x + & 8y + & 4z = & 2 \\ 2x + & 5y + & z = & 5 \\ 4x + & 10y - & z = & 1 \end{array}\right| \begin{array}{l} \div 2 \\ \\ \\ \end{array}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array}\right] \begin{array}{l} \\ -2\,(\text{I}) \\ -4\,(\text{I}) \end{array} \qquad \left|\begin{array}{rrrr} x + & 4y + & 2z = & 1 \\ 2x + & 5y + & z = & 5 \\ 4x + & 10y - & z = & 1 \end{array}\right| \begin{array}{l} \\ -2\,(\text{I}) \\ -4\,(\text{I}) \end{array}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & -3 & -3 & 3 \\ 0 & -6 & -9 & -3 \end{array}\right] \begin{array}{l} \\ \div(-3) \\ \\ \end{array} \qquad \left|\begin{array}{rrrr} x + & 4y + & 2z = & 1 \\ & -3y - & 3z = & 3 \\ & -6y - & 9z = & -3 \end{array}\right| \begin{array}{l} \\ \div(-3) \\ \\ \end{array}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & -6 & -9 & -3 \end{array}\right] \begin{array}{l} -4\,(\text{II}) \\ \\ +6\,(\text{II}) \end{array} \qquad \left|\begin{array}{rrrr} x + & 4y + & 2z = & 1 \\ & y + & z = & -1 \\ & -6y - & 9z = & -3 \end{array}\right| \begin{array}{l} -4\,(\text{II}) \\ \\ +6\,(\text{II}) \end{array}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -3 & -9 \end{array}\right] \begin{array}{l} \\ \\ \div(-3) \end{array} \qquad \left|\begin{array}{rrrr} x & - & 2z = & 5 \\ & y + & z = & -1 \\ & & -3z = & -9 \end{array}\right| \begin{array}{l} \\ \\ \div(-3) \end{array}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} +2\,(\text{III}) \\ -\,(\text{III}) \\ \\ \end{array} \qquad \left|\begin{array}{rrrr} x & - & 2z = & 5 \\ & y + & z = & -1 \\ & & z = & 3 \end{array}\right| \begin{array}{l} +2\,(\text{III}) \\ -\,(\text{III}) \\ \\ \end{array}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

[7] In vector and matrix algebra, the term *scalar* is synonymous with (real) number.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad \left| \begin{array}{rrrl} x & & & = 11 \\ & y & & = -4 \\ & & z & = 3 \end{array} \right|.$$

The solution is often represented as a vector:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \\ 3 \end{bmatrix}.$$

In this example, the process of elimination works very smoothly. We can eliminate all entries off the diagonal and can make each coefficient on the diagonal equal to 1. The process of elimination works well unless we encounter a zero along the diagonal. These zeros represent missing terms in some equations. The following example illustrates how to solve such a system:

$$\left| \begin{array}{rl} x_3 - x_4 - x_5 &= 4 \\ 2x_1 + 4x_2 + 2x_3 + 4x_4 + 2x_5 &= 4 \\ 2x_1 + 4x_2 + 3x_3 + 3x_4 + 3x_5 &= 4 \\ 3x_1 + 6x_2 + 6x_3 + 3x_4 + 6x_5 &= 6 \end{array} \right|.$$

The augmented matrix of this system is

$$M = \left[\begin{array}{ccccc|c} 0 & 0 & 1 & -1 & -1 & 4 \\ 2 & 4 & 2 & 4 & 2 & 4 \\ 2 & 4 & 3 & 3 & 3 & 4 \\ 3 & 6 & 6 & 3 & 6 & 6 \end{array}\right].$$

As in the previous examples, we are trying to bring the matrix into diagonal form. To keep track of our work, we will place a cursor in the matrix, as you might on a computer screen. Initially, the cursor is placed at the top position of the first nonzero column of the matrix:

$$\left[\begin{array}{ccccc|c} \nearrow 0 & 0 & 1 & -1 & -1 & 4 \\ 2 & 4 & 2 & 4 & 2 & 4 \\ 2 & 4 & 3 & 3 & 3 & 4 \\ 3 & 6 & 6 & 3 & 6 & 6 \end{array}\right].$$

Our first goal is to make the cursor entry equal to 1. We can accomplish this in two steps as follows:

Step 1 If the cursor entry is 0, swap the cursor row with some row below to make the cursor entry nonzero.[8]

[8]To make the process unambiguous, swap the cursor row with the *first* row below that has a nonzero entry in the cursor column.

In Step 1, we are merely writing down the equations in a different order. This will certainly not affect the solutions of the system:

$$\left[\begin{array}{ccccc:c} \nearrow 0 & 0 & 1 & -1 & -1 & 4 \\ 2 & 4 & 2 & 4 & 2 & 4 \\ 2 & 4 & 3 & 3 & 3 & 4 \\ 3 & 6 & 6 & 3 & 6 & 6 \end{array}\right] \quad \left|\begin{array}{r} x_3 - x_4 - x_5 = 4 \\ 2x_1 + 4x_2 + 2x_3 + 4x_4 + 2x_5 = 4 \\ 2x_1 + 4x_2 + 3x_3 + 3x_4 + 3x_5 = 4 \\ 3x_1 + 6x_2 + 6x_3 + 3x_4 + 6x_5 = 6 \end{array}\right|$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\left[\begin{array}{ccccc:c} \nearrow 2 & 4 & 2 & 4 & 2 & 4 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 2 & 4 & 3 & 3 & 3 & 4 \\ 3 & 6 & 6 & 3 & 6 & 6 \end{array}\right] \quad \left|\begin{array}{r} 2x_1 + 4x_2 + 2x_3 + 4x_4 + 2x_5 = 4 \\ x_3 - x_4 - x_5 = 4 \\ 2x_1 + 4x_2 + 3x_3 + 3x_4 + 3x_5 = 4 \\ 3x_1 + 6x_2 + 6x_3 + 3x_4 + 6x_5 = 6 \end{array}\right|.$$

Now we can proceed as in the previous examples.

> **Step 2** Divide the cursor row by the cursor entry to make the cursor entry equal to 1.

Step 2 does not change the solutions of the system, because the equation corresponding to the cursor row has the same solutions before and after the operation:

$$\left[\begin{array}{ccccc:c} \nearrow 2 & 4 & 2 & 4 & 2 & 4 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 2 & 4 & 3 & 3 & 3 & 4 \\ 3 & 6 & 6 & 3 & 6 & 6 \end{array}\right] \begin{array}{l} \div 2 \\ \\ \\ \\ \end{array} \quad \left|\begin{array}{r} 2x_1 + 4x_2 + 2x_3 + 4x_4 + 2x_5 = 4 \\ x_3 - x_4 - x_5 = 4 \\ 2x_1 + 4x_2 + 3x_3 + 3x_4 + 3x_5 = 4 \\ 3x_1 + 6x_2 + 6x_3 + 3x_4 + 6x_5 = 6 \end{array}\right| \begin{array}{l} \div 2 \\ \\ \\ \\ \end{array}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\left[\begin{array}{ccccc:c} \nearrow 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 2 & 4 & 3 & 3 & 3 & 4 \\ 3 & 6 & 6 & 3 & 6 & 6 \end{array}\right] \quad \left|\begin{array}{r} x_1 + 2x_2 + x_3 + 2x_4 + x_5 = 2 \\ x_3 - x_4 - x_5 = 4 \\ 2x_1 + 4x_2 + 3x_3 + 3x_4 + 3x_5 = 4 \\ 3x_1 + 6x_2 + 6x_3 + 3x_4 + 6x_5 = 6 \end{array}\right|.$$

> **Step 3** Eliminate all other entries in the cursor column by subtracting suitable multiples of the cursor row from the other rows.[9]

$$\left[\begin{array}{ccccc:c} \nearrow 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 2 & 4 & 3 & 3 & 3 & 4 \\ 3 & 6 & 6 & 3 & 6 & 6 \end{array}\right] \begin{array}{l} \\ \\ -2\,(\mathrm{I}) \\ -3\,(\mathrm{I}) \end{array} \quad \left|\begin{array}{r} x_1 + 2x_2 + x_3 + 2x_4 + x_5 = 2 \\ x_3 - x_4 - x_5 = 4 \\ 2x_1 + 4x_2 + 3x_3 + 3x_4 + 3x_5 = 4 \\ 3x_1 + 6x_2 + 6x_3 + 3x_4 + 6x_5 = 6 \end{array}\right| \begin{array}{l} \\ \\ -2\,(\mathrm{I}) \\ -3\,(\mathrm{I}) \end{array}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\left[\begin{array}{ccccc:c} \nearrow 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 3 & -3 & 3 & 0 \end{array}\right] \quad \left|\begin{array}{r} x_1 + 2x_2 + x_3 + 2x_4 + x_5 = 2 \\ x_3 - x_4 - x_5 = 4 \\ x_3 - x_4 + x_5 = 0 \\ 3x_3 - 3x_4 + 3x_5 = 0 \end{array}\right|$$

---

[9]We may also *add* a multiple of a row, of course. Think of this as subtracting a negative multiple of a row.

Convince yourself that this operation does not change the solutions of the system. (See Exercise 28.)

Now we have taken care of the first column (the first variable), so we can move the cursor to a new position.

Following the approach taken in Section 1.1, we move the cursor down diagonally (i.e., one row down and one column to the right):

$$\left[\begin{array}{ccccc:c} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & \nearrow 0 & 1 & -1 & -1 & 4 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 3 & -3 & 3 & 0 \end{array}\right].$$

For our method to work as before, we need a nonzero cursor entry. Since not only the cursor entry, but also all entries below, are zero, we cannot accomplish this by swapping the cursor row with some row below, as we did in Step 1. It would not help us to swap the cursor row with the row above; this would affect the first column of the matrix, which we have already fixed. Thus, we have to give up on the second column (the second variable); we will move the cursor to the next column:

$$\left[\begin{array}{ccccc:c} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & \nearrow 1 & -1 & -1 & 4 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 3 & -3 & 3 & 0 \end{array}\right].$$

Let's summarize.

> **Step 4** Move the cursor down diagonally (i.e., one row down and one column to the right). If the new cursor entry and all entries below are zero, move the cursor to the next column (remaining in the same row). Repeat this step if necessary. Then return to Step 1.

Here, since the cursor entry is 1, we can proceed directly to Step 3 and eliminate all other entries in the cursor column:

$$\left[\begin{array}{ccccc:c} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & \nearrow 1 & -1 & -1 & 4 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 3 & -3 & 3 & 0 \end{array}\right] \begin{array}{l} -\text{(II)} \\ \\ -\text{(II)} \\ -3\,\text{(II)} \end{array}$$

$$\downarrow \text{Step 3}$$

$$\left[\begin{array}{ccccc:c} 1 & 2 & 0 & 3 & 2 & -2 \\ 0 & 0 & \nearrow 1 & -1 & -1 & 4 \\ 0 & 0 & 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 0 & 6 & -12 \end{array}\right]$$

$$\downarrow \text{Step 4}$$

$$\left[\begin{array}{ccccc:c} 1 & 2 & 0 & 3 & 2 & -2 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 0 & 0 & 0 & 0 & \nearrow 2 & -4 \\ 0 & 0 & 0 & 0 & 6 & -12 \end{array}\right] \begin{array}{l} \\ \\ \div 2 \\ \\ \end{array}$$

$$\downarrow \text{Step 2}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 2 & -2 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 0 & 0 & 0 & 0 & \nearrow 1 & -2 \\ 0 & 0 & 0 & 0 & 6 & -12 \end{array}\right] \begin{array}{l} -2\,(\text{III}) \\ +\,(\text{III}) \\ \\ -6\,(\text{III}) \end{array}$$

$$\downarrow \text{Step 3}$$

$$E = \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & \nearrow 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

When we try to apply Step 4 to this matrix, we run out of columns; the process of row reduction comes to an end. We say that the matrix E is in *reduced row-echelon form*, or *rref* for short. We write

$$E = \text{rref}(M),$$

where M is the augmented matrix of the system.

Reduced row-echelon form

A matrix is in *reduced row-echelon form* if it satisfies all of the following conditions:

a. If a row has nonzero entries, then the first nonzero entry is 1, called the *leading* 1 in this row.[10]

b. If a column contains a leading 1, then all other entries in that column are zero.

c. If a row contains a leading 1, then each row above contains a leading 1 further to the left.

A matrix in reduced row-echelon form may contain rows of zeros, as in the preceding example. However, by condition c, these rows must appear as the last rows of the matrix.

Convince yourself that the procedure just outlined (repeatedly performing Steps 1 through 4) indeed produces a matrix with these three properties. See Exercise 23.

For emphasis, let's circle the leading 1's in the matrix $E = \text{rref}(M)$ we found earlier:

$$E = \left[\begin{array}{ccccc|c} \textcircled{1} & 2 & 0 & 3 & 0 & 2 \\ 0 & 0 & \textcircled{1} & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & \textcircled{1} & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

This matrix represents the following system:

$$\left| \begin{array}{rcrcrcr} \textcircled{x_1} + 2x_2 & & + 3x_4 & & & = & 2 \\ & & \textcircled{x_3} - x_4 & & & = & 2 \\ & & & & \textcircled{x_5} & = & -2 \end{array} \right|.$$

Again, we say that this system is in reduced row-echelon form. The *leading variables* correspond to the leading 1's in the echelon form of the matrix. We also

[10] A leading 1 is alternatively referred to as a *pivot*.

draw the staircase formed by the leading variables. That is where the name *echelon form* comes from: According to Webster, an echelon is a formation "like a series of steps."

Now we can solve each of the preceding equations for the leading variable:

$$\begin{aligned} x_1 &= 2 - 2x_2 - 3x_4 \\ x_3 &= 2 + x_4 \\ x_5 &= -2. \end{aligned}$$

We can freely choose the nonleading variables, $x_2 = s$ and $x_4 = t$, where s and t are arbitrary real numbers; the nonleading variables are alternatively referred to as the *free* variables. The leading variables are then determined by our choices for s and t; that is, $x_1 = 2 - 2s - 3t$, $x_3 = 2 + t$, and $x_5 = -2$.

This system has infinitely many solutions, namely,

$$x_1 = 2 - 2s - 3t, \quad x_2 = s, \quad x_3 = 2 + t, \quad x_4 = t, \quad x_5 = -2,$$

where s and t are arbitrary real numbers. We can represent the solutions as vectors in $\mathbb{R}^5$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 - 2s - 3t \\ s \\ 2 + t \\ t \\ -2 \end{bmatrix} \qquad (s, t \text{ arbitrary}).$$

We will often find it helpful to write this solution as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \\ -2 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

For example, if we set $s = t = 0$, we get the particular solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \\ -2 \end{bmatrix}.$$

Here is a summary of the elimination process just outlined:

Solving systems of linear equations

Write the augmented matrix of the system. Place a cursor in the top entry of the first nonzero column of this matrix.

Step 1 If the cursor entry is zero, swap the cursor row with some row below to make the cursor entry nonzero.

Step 2 Divide the cursor row by the cursor entry.

Step 3 Eliminate all other entries in the cursor column, by subtracting suitable multiples of the cursor row from the other rows.

Step 4 Move the cursor one row down and one column to the right. If the new cursor entry and all entries below are zero, move the cursor to the next column (remaining in the same row). Repeat the last step if necessary.

Return to Step 1.

The process ends when we run out of rows or columns. Then, the matrix is in reduced row-echelon form (rref).

Write down the linear system corresponding to this matrix, and solve each equation in the system for the leading variable. You may choose the nonleading variables freely; the leading variables are then determined by these choices. If the echelon form contains the equation $0 = 1$, then there are no solutions; the system is *inconsistent*.

The operations performed in Steps 1, 2, and 3 are called *elementary row operations*: Swap two rows, divide a row by a scalar, or subtract a multiple of a row from another row.

The following is an inconsistent system:

$$\left| \begin{array}{rcrcrcrcr} x_1 & - & 3x_2 & & & - & 5x_4 & = & -7 \\ 3x_1 & - & 12x_2 & - & 2x_3 & - & 27x_4 & = & -33 \\ -2x_1 & + & 10x_2 & + & 2x_3 & + & 24x_4 & = & 29 \\ -x_1 & + & 6x_2 & + & x_3 & + & 14x_4 & = & 17 \end{array} \right| .$$

The augmented matrix of the system is

$$\left[\begin{array}{rrrr|r} 1 & -3 & 0 & -5 & -7 \\ 3 & -12 & -2 & -27 & -33 \\ -2 & 10 & 2 & 24 & 29 \\ -1 & 6 & 1 & 14 & 17 \end{array} \right].$$

The reduced row-echelon form for this matrix is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right].$$

(We leave it to you to perform the elimination.)

Since the last row of the echelon form represents the equation $0 = 1$, the system is inconsistent.

This method of solving linear systems is sometimes referred to as *Gauss–Jordan elimination*, after the German mathematician Carl Friedrich Gauss (1777–1855; see Figure 4), perhaps the greatest mathematician of modern times, and the German engineer Wilhelm Jordan (1844–1899). Gauss himself called the method *eliminatio vulgaris*. Recall that the Chinese were using this method 2,000 years ago.

Figure 4 Carl Friedrich Gauss appears on an old German 10-mark note. (In fact, this is the *mirror image* of a well-known portrait of Gauss.[11])

How Gauss developed this method is noteworthy. On January 1, 1801, the Sicilian astronomer Giuseppe Piazzi (1746–1826) discovered a planet, which he named "Ceres," in honor of the patron goddess of Sicily. Today, Ceres is called an asteroid or minor planet; it is only about 1,000 km in diameter. The public was very interested in this discovery. At that time, the number of planets in the solar system was still an issue debated by many philosophers and representatives of the Church. Piazzi was able to observe Ceres for forty nights, but then he lost track of it. Gauss, however, at the age of 24, succeeded in calculating the orbit of Ceres, even though the task seemed hopeless on the basis of a few observations. His computations were so accurate that the German astronomer W. Olbers (1758–1840) located the asteroid on December 31, 1801. In the course of his computations, Gauss had to solve systems of 17 linear equations. In dealing with this problem, Gauss also used the method of least squares, which he had developed around 1794. (See Section 5.4.) Since Gauss at first refused to reveal the methods that led to this amazing accomplishment, some even accused him of sorcery. Gauss later described his methods of orbit computation in his book *Theoria Motus Corporum Coelestium* (1809).

The method of solving a linear system by Gauss–Jordan elimination is called an *algorithm*.[12] An algorithm can be defined as "a finite procedure, written in a fixed symbolic vocabulary, governed by precise instructions, moving in discrete Steps, 1, 2, 3, ... , whose execution requires no insight, cleverness, intuition, intelligence, or perspicuity, and that sooner or later comes to an end" (David Berlinski, *The Advent of the Algorithm: The Idea that Rules the World*, Harcourt Inc., 2000).

Gauss–Jordan elimination is well suited for solving linear systems on a computer, at least in principle. In practice, however, some tricky problems associated with roundoff errors can occur.

Numerical analysis tells us that we can reduce the proliferation of roundoff errors by modifying Gauss–Jordan elimination with partial or complete *pivoting* techniques. Partial pivoting requires us to modify Step 1 of the algorithm as follows: Swap the cursor row with a row below to make the cursor entry as large as

[11] Reproduced by permission of the German Bundesbank.

[12] The word *algorithm* is derived from the name of the mathematician al-Khowarizmi, who introduced the term *algebra* into mathematics. (See page 1.)

possible (in absolute value). This swap is performed even if the initial cursor entry is nonzero, as long as there is an entry below with a larger absolute value. In the first example worked in the text, we would start by swapping the first row with the last:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 39 \\ 1 & 3 & 2 & 34 \\ 3 & 2 & 1 & 26 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 3 & 2 & 1 & 26 \\ 1 & 3 & 2 & 34 \\ 1 & 2 & 3 & 39 \end{array}\right].$$

In modifying Gauss–Jordan elimination, an interesting question arises: If we transform a matrix A into a matrix B by a sequence of elementary row operations and if B is in reduced row-echelon form, is it necessarily true that $B = \text{rref}(A)$? Fortunately (and perhaps surprisingly) this is indeed the case.

In this text, we will not utilize this fact, so there is no need to present the somewhat technical proof. If you feel ambitious, try to work out the proof yourself after studying Chapter 3. (See Exercises 3.3.64 through 3.3.67.)

EXERCISES 1.2

GOAL Use Gauss–Jordan elimination to solve linear systems. Do simple problems using paper and pencil, and use technology to solve more complicated problems.

In Exercises 1 through 12, find all solutions of the equations with paper and pencil using Gauss–Jordan elimination. Show all your work. Solve the system in Exercise 8 for the variables x_1, x_2, x_3, x_4, and x_5.

1. $\left|\begin{array}{l} x + y - 2z = 5 \\ 2x + 3y + 4z = 2 \end{array}\right|$

2. $\left|\begin{array}{l} 3x + 4y - z = 8 \\ 6x + 8y - 2z = 3 \end{array}\right|$

3. $x + 2y + 3z = 4$

4. $\left|\begin{array}{l} x + y = 1 \\ 2x - y = 5 \\ 3x + 4y = 2 \end{array}\right|$

5. $\left|\begin{array}{l} x_3 + x_4 = 0 \\ x_2 + x_3 = 0 \\ x_1 + x_2 = 0 \\ x_1 + x_4 = 0 \end{array}\right|$

6. $\left|\begin{array}{l} x_1 - 7x_2 + x_5 = 3 \\ x_3 - 2x_5 = 2 \\ x_4 + x_5 = 1 \end{array}\right|$

7. $\left|\begin{array}{l} x_1 + 2x_2 \quad 2x_4 + 3x_5 = 0 \\ x_3 + 3x_4 + 2x_5 = 0 \\ x_3 + 4x_4 - x_5 = 0 \\ x_5 = 0 \end{array}\right|$

8. $\left|\begin{array}{l} x_2 + 2x_4 + 3x_5 = 0 \\ 4x_4 + 8x_5 = 0 \end{array}\right|$

9. $\left|\begin{array}{l} x_4 + 2x_5 - x_6 = 2 \\ x_1 + 2x_2 + x_5 - x_6 = 0 \\ x_1 + 2x_2 + 2x_3 - x_5 + x_6 = 2 \end{array}\right|$

10. $\left|\begin{array}{l} 4x_1 + 3x_2 + 2x_3 - x_4 = 4 \\ 5x_1 + 4x_2 + 3x_3 - x_4 = 4 \\ -2x_1 - 2x_2 - x_3 + 2x_4 = -3 \\ 11x_1 + 6x_2 + 4x_3 + x_4 = 11 \end{array}\right|$

11. $\left|\begin{array}{l} x_1 + 2x_3 + 4x_4 = -8 \\ x_2 - 3x_3 - x_4 = 6 \\ 3x_1 + 4x_2 - 6x_3 + 8x_4 = 0 \\ -x_2 + 3x_3 + 4x_4 = -12 \end{array}\right|$

12. $\left|\begin{array}{l} 2x_1 - 3x_3 + 7x_5 + 7x_6 = 0 \\ -2x_1 + x_2 + 6x_3 - 6x_5 - 12x_6 = 0 \\ x_2 - 3x_3 + x_5 + 5x_6 = 0 \\ -2x_2 + x_4 + x_5 + x_6 = 0 \\ 2x_1 + x_2 - 3x_3 + 8x_5 + 7x_6 = 0 \end{array}\right|$

Solve the linear systems in Exercises 13 through 17. You may use technology.

13. $\left|\begin{array}{l} 3x + 11y + 19z = -2 \\ 7x + 23y + 39z = 10 \\ -4x - 3y - 2z = 6 \end{array}\right|$

14. $\left|\begin{array}{l} 3x + 6y + 14z = 22 \\ 7x + 14y + 30z = 46 \\ 4x + 8y + 7z = 6 \end{array}\right|$

15. $\left|\begin{array}{l} 3x + 5y + 3z = 25 \\ 7x + 9y + 19z = 65 \\ -4x + 5y + 11z = 5 \end{array}\right|$

16. $\left|\begin{array}{l} 3x_1 + 6x_2 + 9x_3 + 5x_4 + 25x_5 = 53 \\ 7x_1 + 14x_2 + 21x_3 + 9x_4 + 53x_5 = 105 \\ -4x_1 - 8x_2 - 12x_3 + 5x_4 - 10x_5 = 11 \end{array}\right|$

17. $\left|\begin{array}{l} 2x_1 + 4x_2 + 3x_3 + 5x_4 + 6x_5 = 37 \\ 4x_1 + 8x_2 + 7x_3 + 5x_4 + 2x_5 = 74 \\ -2x_1 - 4x_2 + 3x_3 + 4x_4 - 5x_5 = 20 \\ x_1 + 2x_2 + 2x_3 - x_4 + 2x_5 = 26 \\ 5x_1 - 10x_2 + 4x_3 + 6x_4 + 4x_5 = 24 \end{array}\right|$

18. Determine which of the matrices below are in reduced row-echelon form:

a. $\begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ d. $\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \end{bmatrix}$

19. Find all 4×1 matrices in reduced row-echelon form.

20. We say that two $n \times m$ matrices in reduced row-echelon form are of the same type if they contain the same number of leading 1's in the same positions. For example,

$$\begin{bmatrix} \textcircled{1} & 2 & 0 \\ 0 & 0 & \textcircled{1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \textcircled{1} & 3 & 0 \\ 0 & 0 & \textcircled{1} \end{bmatrix}$$

are of the same type. How many types of 2×2 matrices in reduced row-echelon form are there?

21. How many types of 3×2 matrices in reduced row-echelon form are there? (See Exercise 20.)

22. How many types of 2×3 matrices in reduced row-echelon form are there? (See Exercise 20.)

23. Suppose you apply Gauss–Jordan elimination to a matrix. Explain how you can be sure that the resulting matrix is in reduced row-echelon form.

24. Suppose matrix A is transformed into matrix B by means of an elementary row operation. Is there an elementary row operation that transforms B into A? Explain.

25. Suppose matrix A is transformed into matrix B by a sequence of elementary row operations. Is there a sequence of elementary row operations that transforms B into A? Explain your answer. (See Exercise 24.)

26. Consider an $n \times m$ matrix A. Can you transform rref(A) into A by a sequence of elementary row operations? (See Exercise 25.)

27. Is there a sequence of elementary row operations that transforms

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{into} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}?$$

Explain.

28. Suppose you subtract a multiple of an equation in a system from another equation in the system. Explain why the two systems (before and after this operation) have the same solutions.

29. ***Balancing a Chemical Reaction.*** Consider the chemical reaction

$$a\,NO_2 + b\,H_2O \rightarrow c\,HNO_2 + d\,HNO_3,$$

where a, b, c, and d are unknown positive integers. The reaction must be balanced; that is, the number of atoms of each element must be the same before and after the reaction. For example, because the number of oxygen atoms must remain the same,

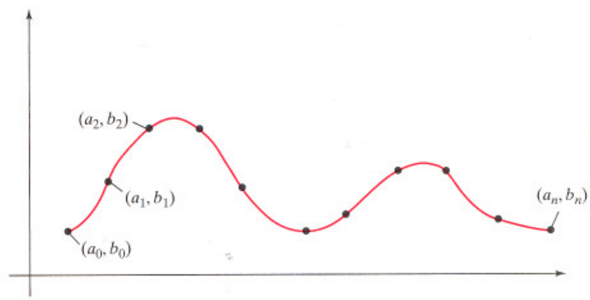
$$2a + b = 2c + 3d.$$

While there are many possible values for a, b, c, and d that balance the reaction, it is customary to use the smallest possible positive integers. Balance this reaction.

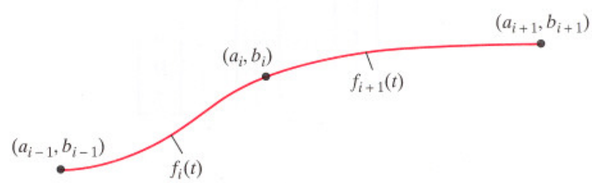
30. Find the polynomial of degree 3 [a polynomial of the form $f(t) = a + bt + ct^2 + dt^3$] whose graph goes through the points (0, 1), (1, 0), (−1, 0), and (2, −15). Sketch the graph of this cubic.

31. Find the polynomial of degree 4 whose graph goes through the points (1, 1), (2, −1), (3, −59), (−1, 5), and (−2, −29). Graph this polynomial.

32. ***Cubic Splines.*** Suppose you are in charge of the design of a roller coaster ride. This simple ride will not make any left or right turns; that is, the track lies in a vertical plane. The accompanying figure shows the ride as viewed from the side. The points (a_i, b_i) are given to you, and your job is to connect the dots in a reasonably smooth way. Let $a_{i+1} > a_i$.



One method often employed in such design problems is the technique of *cubic splines*. We choose $f_i(t)$, a polynomial of degree ≤ 3, to define the shape of the ride between (a_{i-1}, b_{i-1}) and (a_i, b_i), for $i = 1, \ldots, n$.



Obviously, it is required that $f_i(a_i) = b_i$ and $f_i(a_{i-1}) = b_{i-1}$, for $i = 1, \ldots, n$. To guarantee a smooth ride at the points (a_i, b_i), we want the first and the second derivatives of f_i and f_{i+1} to agree at these points:

$$f_i'(a_i) = f_{i+1}'(a_i) \quad \text{and}$$
$$f_i''(a_i) = f_{i+1}''(a_i), \quad \text{for } i = 1, \ldots, n-1.$$

Explain the practical significance of these conditions. Explain why, for the convenience of the riders, it is also required that

$$f_1'(a_0) = f_n'(a_n) = 0.$$

Show that satisfying all these conditions amounts to solving a system of linear equations. How many variables are in this system? How many equations? (*Note*: It can be shown that this system has a unique solution.)

33. Find the polynomial $f(t)$ of degree 3 such that $f(1) = 1$, $f(2) = 5$, $f'(1) = 2$, and $f'(2) = 9$, where $f'(t)$ is the derivative of $f(t)$. Graph this polynomial.

34. The *dot product* of two vectors

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

in $\mathbb{R}^n$ is defined by

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Note that the dot product of two vectors is a scalar. We say that the vectors $\vec{x}$ and $\vec{y}$ are *perpendicular* if $\vec{x} \cdot \vec{y} = 0$.

Find all vectors in $\mathbb{R}^3$ perpendicular to

$$\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}.$$

Draw a sketch.

35. Find all vectors in $\mathbb{R}^4$ that are perpendicular to the three vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 9 \\ 9 \\ 7 \end{bmatrix}.$$

(See Exercise 34.)

36. Find all solutions x_1, x_2, x_3 of the equation

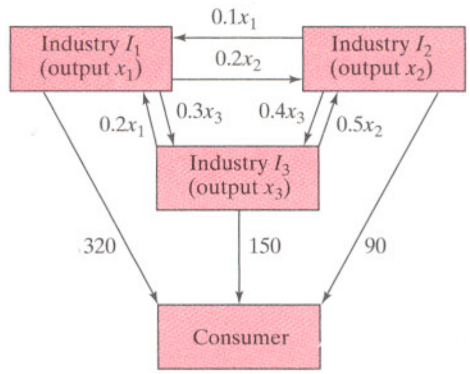
$$\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3,$$

where

$$\vec{b} = \begin{bmatrix} -8 \\ -1 \\ 2 \\ 15 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \\ 5 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 8 \\ 3 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 4 \\ 6 \\ 9 \\ 1 \end{bmatrix}.$$

37. For some background on this exercise, see Exercise 1.1.20.

Consider an economy with three industries, I_1, I_2, I_3. What outputs x_1, x_2, x_3 should they produce to satisfy both consumer demand and interindustry demand? The demands put on the three industries are shown in the accompanying figure.



38. If we consider more than three industries in an input–output model, it is cumbersome to represent all the demands in a diagram as in Exercise 37. Suppose we have the industries $I_1, I_2, \ldots, I_n$, with outputs $x_1, x_2, \ldots, x_n$. The *output vector* is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

The *consumer demand vector* is

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

where b_i is the consumer demand on industry I_i. The demand vector for industry I_j is

$$\vec{v}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix},$$

where a_{ij} is the demand industry I_j puts on industry I_i, for each \$1 of output industry I_j produces. For example, $a_{32} = 0.5$ means that industry I_2 needs 50¢ worth of products from industry I_3 for each \$1 worth of goods I_2 produces. The coefficient a_{ii} need not be 0: Producing a product may require goods or services from the same industry.

a. Find the four demand vectors for the economy in Exercise 37.

b. What is the meaning in economic terms of $x_j\vec{v}_j$?

c. What is the meaning in economic terms of $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n + \vec{b}$?

d. What is the meaning in economic terms of the equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n + \vec{b} = \vec{x}?$$

39. Consider the economy of Israel in 1958.[13] The three industries considered here are

I_1 : agriculture,
I_2 : manufacturing,
I_3 : energy.

Outputs and demands are measured in millions of Israeli pounds, the currency of Israel at that time. We are told that

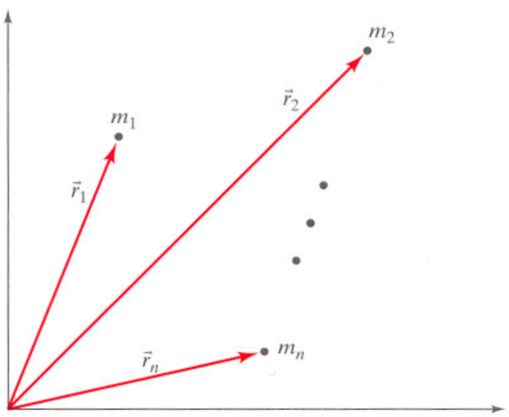
$$\vec{b} = \begin{bmatrix} 13.2 \\ 17.6 \\ 1.8 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 0.293 \\ 0.014 \\ 0.044 \end{bmatrix},$$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 0.207 \\ 0.01 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 0.017 \\ 0.216 \end{bmatrix}.$$

a. Why do the first components of $\vec{v}_2$ and $\vec{v}_3$ equal 0?

b. Find the outputs x_1, x_2, x_3 required to satisfy demand.

40. Consider some particles in the plane with position vectors $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n$ and masses $m_1, m_2, \ldots, m_n$.

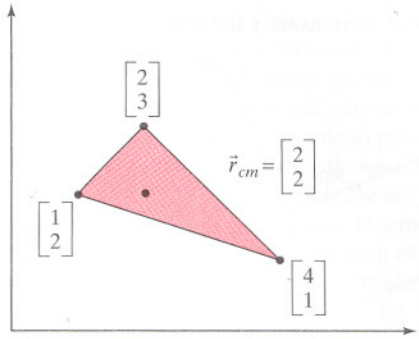


The position vector of the *center of mass* of this system is

$$\vec{r}_{cm} = \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_n\vec{r}_n),$$

where $M = m_1 + m_2 + \cdots + m_n$.

Consider the triangular plate shown in the accompanying sketch. How must a total mass of 1 kg be distributed among the three vertices of the plate so that the plate can be supported at the point $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$; that is, $\vec{r}_{cm} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$? Assume that the mass of the plate itself is negligible.



41. The *momentum* $\vec{P}$ of a system of n particles in space with masses $m_1, m_2, \ldots, m_n$ and velocities $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is defined as

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots + m_n\vec{v}_n.$$

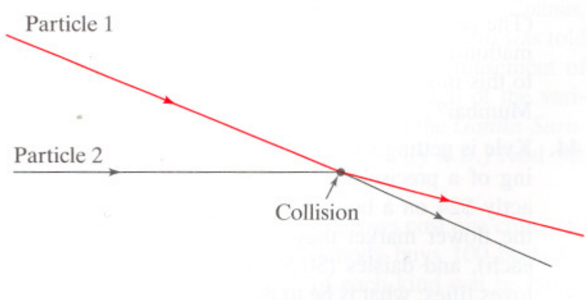
Now consider two elementary particles with velocities

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix}.$$

The particles collide. After the collision, their respective velocities are observed to be

$$\vec{w}_1 = \begin{bmatrix} 4 \\ 7 \\ 4 \end{bmatrix} \quad \text{and} \quad \vec{w}_2 = \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}.$$

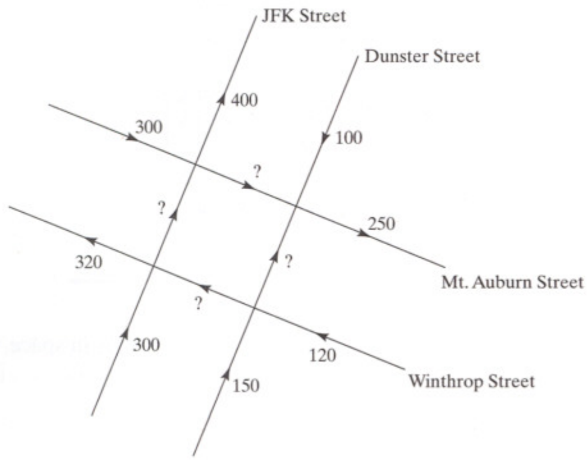
Assume that the momentum of the system is conserved throughout the collision. What does this experiment tell you about the masses of the two particles? (See the accompanying figure.)



42. The accompanying sketch represents a maze of one-way streets in a city in the United States. The traffic volume through certain blocks during an hour has been measured. Suppose that the vehicles leaving the area during

[13]W. Leontief: *Input–Output Economics*, Oxford University Press, 1966.

this hour were exactly the same as those entering it.



What can you say about the traffic volume at the four locations indicated by a question mark? Can you figure out exactly how much traffic there was on each block? If not, describe one possible scenario. For each of the four locations, find the highest and the lowest possible traffic volume.

43. Let $S(t)$ be the length of the tth day of the year in Mumbai (formerly known as Bombay), India (measured in hours, from sunrise to sunset). We are given the following values of $S(t)$:

t	$S(t)$
47	11.5
74	12
273	12

For example, $S(47) = 11.5$ means that the time from sunrise to sunset on February 16 is 11 hours and 30 minutes. For locations close to the equator, the function $S(t)$ is well approximated by a trigonometric function of the form

$$S(t) = a + b\cos\left(\frac{2\pi t}{365}\right) + c\sin\left(\frac{2\pi t}{365}\right).$$

(The period is 365 days, or 1 year.) Find this approximation for Mumbai, and graph your solution. According to this model, how long is the longest day of the year in Mumbai?

44. Kyle is getting some flowers for Kate, his Valentine. Being of a precise analytical mind, he plans to spend exactly \$24 on a bunch of exactly two dozen flowers. At the flower market they have lilies (\$3 each), roses (\$2 each), and daisies (\$0.50 each). Kyle knows that Kate loves lilies; what is he to do?

45. Consider the equations

$$\left| \begin{array}{l} x + 2y + \quad\quad 3z = 4 \\ x + ky + \quad\quad 4z = 6 \\ x + 2y + (k+2)z = 6 \end{array} \right|,$$

where k is an arbitrary constant.

a. For which values of the constant k does this system have a unique solution?

b. When is there no solution?

c. When are there infinitely many solutions?

46. Consider the equations

$$\left| \begin{array}{r} y + 2kz = 0 \\ x + 2y + \quad 6z = 2 \\ kx \quad\quad + \quad 2z = 1 \end{array} \right|,$$

where k is an arbitrary constant.

a. For which values of the constant k does this system have a unique solution?

b. When is there no solution?

c. When are there infinitely many solutions?

47. a. Find all solutions x_1, x_2, x_3, x_4 of the system $x_2 = \frac{1}{2}(x_1 + x_3)$, $x_3 = \frac{1}{2}(x_2 + x_4)$.

b. In part (a), is there a solution with $x_1 = 1$ and $x_4 = 13$?

48. For an arbitrary positive integer $n \geq 3$, find all solutions $x_1, x_2, x_3, \ldots, x_n$ of the simultaneous equations $x_2 = \frac{1}{2}(x_1 + x_3)$, $x_3 = \frac{1}{2}(x_2 + x_4)$, $\ldots$, $x_{n-1} = \frac{1}{2}(x_{n-2} + x_n)$. Note that we are asked to solve the simultaneous equations $x_k = \frac{1}{2}(x_{k-1} + x_{k+1})$, for $k = 2, 3, \ldots, n-1$.

49. Consider the system

$$\left| \begin{array}{l} 2x + \; y \quad\quad\; = C \\ \quad\quad 3y + \; z = C \\ x \quad\quad\; + 4z = C \end{array} \right|,$$

where C is a constant. Find the smallest positive integer C such that x, y, and z are all integers.

50. Find all the polynomials $f(t)$ of degree ≤ 3 such that $f(0) = 3$, $f(1) = 2$, $f(2) = 0$, and $\int_0^2 f(t)\,dt = 4$. (If you have studied Simpson's rule in calculus, explain the result.)

51. Students are buying books for the new semester. Eddie buys the environmental statistics book and the set theory book for \$178. Leah, who is buying books for herself and her friend, spends \$319 on two environmental statistics books, one set theory book, and one educational psychology book. Mehmet buys the educational psychology book and the set theory book for \$147 in total. How much does each book cost?

52. Students are buying books for the new semester. Brigitte buys the German grammar book and the German novel, *Die Leiden des jungen Werther*, for €64 in total. Claude spends €98 on the linear algebra text and the German grammar book, while Denise buys the linear algebra text and *Werther*, for €76. How much does each of the three books cost?

53. At the beginning of a political science class at a large university, the students were asked which term, *liberal* or *conservative*, best described their political views. They were asked the same question at the end of the course, to see what effect the class discussions had on their views. Of those that characterized themselves as "liberal" initially, 30% held conservative views at the end. Of those who were conservative initially, 40% moved to the liberal camp. It turned out that there were just as many students with conservative views at the end as there had been liberal students at the beginning. Out of the 260 students in the class, how many held liberal and conservative views at the beginning of the course and at the end? (No students joined or dropped the class between the surveys, and they all participated in both surveys.)

54. At the beginning of a semester, 55 students have signed up for Linear Algebra; the course is offered in two sections that are taught at different times. Because of scheduling conflicts and personal preferences, 20% of the students in Section A switch to Section B in the first few weeks of class, while 30% of the students in Section B switch to A, resulting in a net loss of 4 students for Section B. How large were the two sections at the beginning of the semester? No students dropped Linear Algebra (why would they?) or joined the course late.

Historical Problems

55. Five cows and two sheep together cost ten *liang*[14] of silver. Two cows and five sheep together cost eight *liang* of silver. What is the cost of a cow and a sheep, respectively? (*Nine Chapters*,[15] Chapter 8, Problem 7)

56. If you sell two cows and five sheep and you buy 13 pigs, you gain 1,000 coins. If you sell three cows and three pigs and buy nine sheep, you break even. If you sell six sheep and eight pigs and you buy five cows, you lose 600 coins. What is the price of a cow, a sheep, and a pig, respectively? (*Nine Chapters*, Chapter 8, Problem 8)

57. You place five sparrows on one of the pans of a balance and six swallows on the other pan; it turns out that the sparrows are heavier. But if you exchange one sparrow and one swallow, the weights are exactly balanced. All the birds together weigh 1 *jin*. What is the weight of a sparrow and a swallow, respectively? [Give the answer in *liang*, with 1 *jin* = 16 *liang*]. (*Nine Chapters*, Chapter 8, Problem 9)

58. Consider the task of pulling a weight of 40 *dan*[16] up a hill; we have one military horse, two ordinary horses, and three weak horses at our disposal to get the job done. It turns out that the military horse and one of the ordinary horses, pulling together, are barely able to pull the weight (but they could not pull any more). Likewise, the two ordinary horses together with one weak horse are just able to do the job, as are the three weak horses together with the military horse. How much weight can each of the horses pull alone? (*Nine Chapters*, Chapter 8, Problem 12)

59. Five households share a deep well for their water supply. Each household owns a few ropes of a certain length, which varies only from household to household. The five households, A, B, C , D , and E, own 2, 3, 4, 5, and 6 ropes, respectively. Even when tying all their ropes together, none of the households alone is able to reach the water, but A's two ropes together with one of B's ropes just reach the water. Likewise, B's three ropes with one of C's ropes, C's four ropes with one of D's ropes, D's five ropes with one of E's ropes, and E's six ropes with one of A's ropes all just reach the water. How long are the ropes of the various households, and how deep is the well?
Commentary: As stated, this problem leads to a system of 5 linear equations in 6 variables; with the given information, we are unable to determine the depth of the well. The *Nine Chapters* give one particular solution, where the depth of the well is 7 *zhang*,[17] 2 *chi*, 1 *cun*, or 721 *cun* (since 1 *zhang* = 10 *chi* and 1 *chi* = 10 *cun*). Using this particular value for the depth of the well, find the lengths of the various ropes.

60. "A rooster is worth five coins, a hen three coins, and 3 chicks one coin. With 100 coins we buy 100 of them. How many roosters, hens, and chicks can we buy?" (From the *Mathematical Manual* by Zhang Qiujian, Chapter 3, Problem 38; 5th century A.D.)
Commentary: This famous *Hundred Fowl Problem* has reappeared in countless variations in Indian, Arabic, and European texts (see Exercises 61 through 64); it has remained popular to this day (see Exercise 44 of this section).

61. "Pigeons are sold at the rate of 5 for 3 panas, sarasabirds at the rate of 7 for 5 panas, swans at the rate of 9 for 7 panas, and peacocks at the rate of 3 for 9 panas. A man was told to bring 100 birds for 100 panas for the amusement of the King's son. What does he pay for each of the various kinds of birds that he buys?" (From the *Ganita-Sara-Sangraha* by Mahavira, India; 9th century A.D.) Find one solution to this problem.

62. "A duck costs four coins, five sparrows cost one coin, and a rooster costs one coin. Somebody buys 100 birds for 100 coins. How many birds of each kind can he buy?" (From the *Key to Arithmetic* by Al-Kashi; 15th century)

[14] A *liang* was about 16 grams at the time of the Han Dynasty.

[15] See page 1; we present some of the problems from the *Nine Chapters on the Mathematical Art* in a free translation, with some additional explanations, since the scenarios discussed in a few of these problems are rather unfamiliar to the modern reader.

[16] 1 *dan* = 120 *jin* = 1920 *liang*. Thus a *dan* was about 30 kg at that time.

[17] 1 *zhang* was about 2.3 meters at that time.

63. "A certain person buys sheep, goats, and hogs, to the number of 100, for 100 crowns; the sheep cost him $\frac{1}{2}$ a crown a-piece; the goats, $1\frac{1}{3}$ crown; and the hogs $3\frac{1}{2}$ crowns. How many had he of each?" (From the *Elements of Algebra* by Leonhard Euler, 1770)

64. "A gentleman has a household of 100 persons and orders that they be given 100 measures of grain. He directs that each man should receive three measures, each woman two measures, and each child half a measure. How many men, women, and children are there in this household?" We are told that there is at least one man, one woman, and one child. (From the *Problems for Quickening a Young Mind* by Alcuin [c.732–804], the Abbot of St. Martins at Tours. Alcuin was a friend and tutor to Charlemagne and his family at Aachen.)

65. A father, when dying, gave to his sons 30 barrels, of which 10 were full of wine, 10 were half full, and the last 10 were empty. Divide the wine and flasks so that there will be equal division among the three sons of both wine and barrels. Find all the solutions of this problem. (From Alcuin)

66. "Make me a crown weighing 60 *minae*, mixing gold, bronze, tin, and wrought iron. Let the gold and bronze together form two-thirds, the gold and tin together three-fourths, and the gold and iron three-fifths. Tell me how much gold, tin, bronze, and iron you must put in." (From the *Greek Anthology* by Metrodorus, 6th century A.D.)

67. Three merchants find a purse lying in the road. One merchant says "If I keep the purse, I shall have twice as much money as the two of you together." "Give me the purse and I shall have three times as much as the two of you together" said the second merchant. The third merchant said "I shall be much better off than either of you if I keep the purse, I shall have five times as much as the two of you together." If there are 60 coins (of equal value) in the purse, how much money does each merchant have? (From Mahavira)

68. 3 cows graze 1 field bare in 2 days,
7 cows graze 4 fields bare in 4 days, and
3 cows graze 2 fields bare in 5 days.
It is assumed that each field initially provides the same amount, x, of grass; that the daily growth, y, of the fields remains constant; and that all the cows eat the same amount, z, each day. (Quantities x, y, and z are measured by weight.) Find all the solutions of this problem. (This is a special case of a problem discussed by Isaac Newton in his *Arithmetica Universalis*, 1707.)

1.3 ON THE SOLUTIONS OF LINEAR SYSTEMS; MATRIX ALGEBRA

In this final section of Chapter 1, we will discuss two rather unrelated topics:

- First, we will examine how many solutions a system of linear equations can possibly have.
- Then, we will present some definitions and rules of matrix algebra.

The Number of Solutions of a Linear System

EXAMPLE 1 The reduced row-echelon forms of the augmented matrices of three systems are given. How many solutions are there in each case?

a. $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ b. $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ c. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

Solution

a. The third row represents the equation $0 = 1$, so that there are no solutions. We say that this system is *inconsistent*.

b. The given augmented matrix represents the system

$$\left| \begin{array}{rr} x_1 + 2x_2 & = 1 \\ x_3 & = 2 \end{array} \right|, \quad \text{or,} \quad \left| \begin{array}{l} x_1 = 1 - 2x_2 \\ x_3 = 2 \end{array} \right|.$$

We can assign an arbitrary value, t, to the free variable x_2, so that the system has