Rotation Examples

Gene Quinn

Rotations

A **rotation** is a type of linear transformation

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that rotates a vector through an angle θ.

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Like all linear transformations, a rotation is equivalent to multiplication by some matrix A:

$$T(\vec{v}) \; = \; A\vec{v}$$

for some matrix A.

Rotations

In the special case of two dimensions,

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

the matrix A of a rotation has the form

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

where $a^2 + b^2 = 1$.

Rotations

Consider the linear transformation that rotates a vector through an angle of $\pi/2$ (90 degrees):

$$\cos\frac{\pi}{2} = 0 \quad \text{and} \quad \sin\frac{\pi}{2} = 1$$

so that

$$A = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

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So

$$T\vec{x} = A\vec{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

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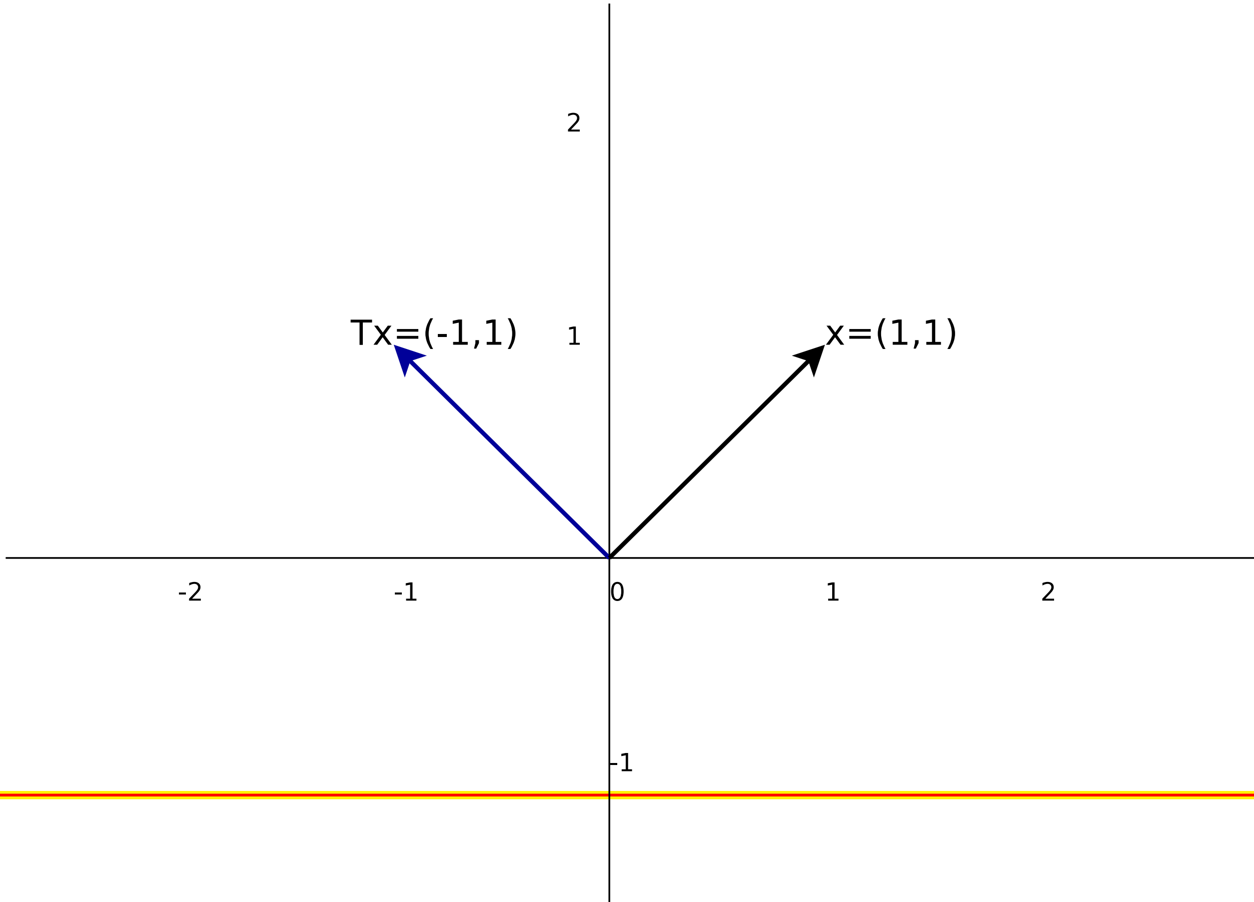
Evidently the effect of this transformation is to interchange x_1 and x_2, and reverse the sign of x_2.

Rotations

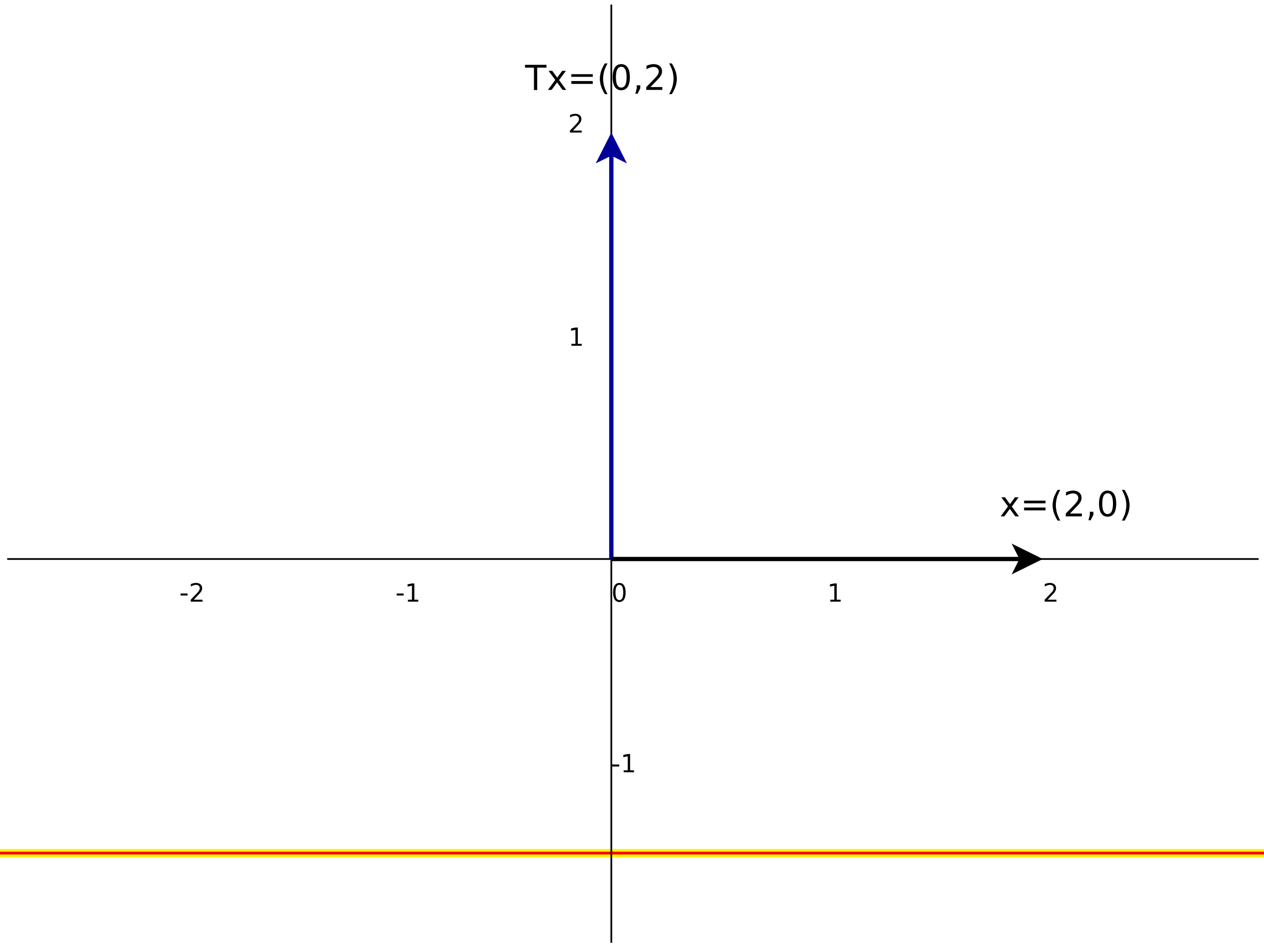
Let's examine the action of T on some vectors in $\mathbb{R}^2$:

$$\text{let } \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ then } T(\vec{x}) = A\vec{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



Rotations

let $\vec{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ then $T(\vec{x}) = A\vec{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$



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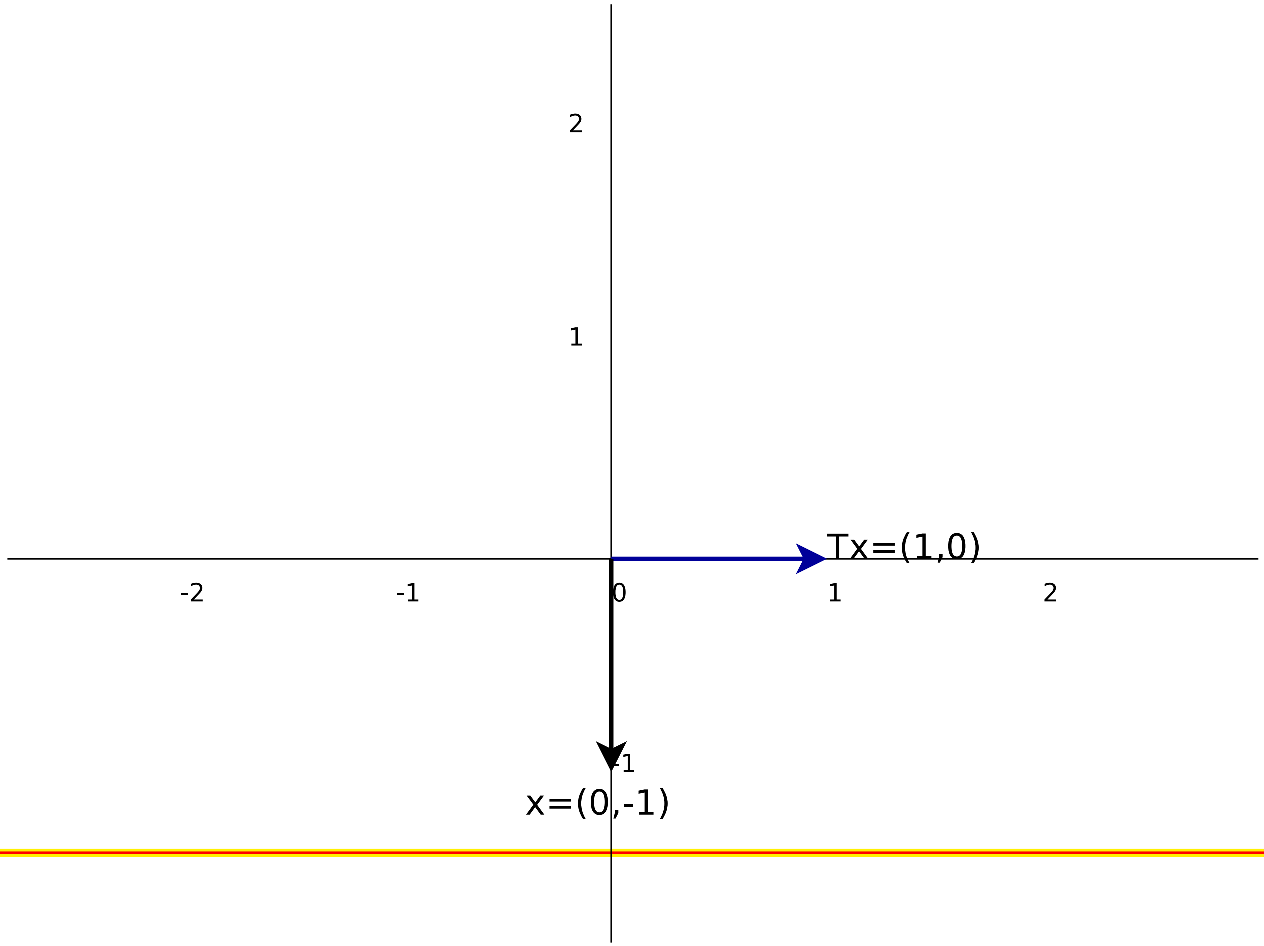

Rotations

Consider the linear transformation that rotates a vector through an angle of $3\pi/2$ (270 degrees):

$$\cos\frac{3\pi}{2} = 0 \quad \text{and} \quad \sin\frac{3\pi}{2} = -1$$

so that

$$A = \begin{bmatrix} \cos\frac{3\pi}{2} & -\sin\frac{3\pi}{2} \\ \sin\frac{3\pi}{2} & \cos\frac{3\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

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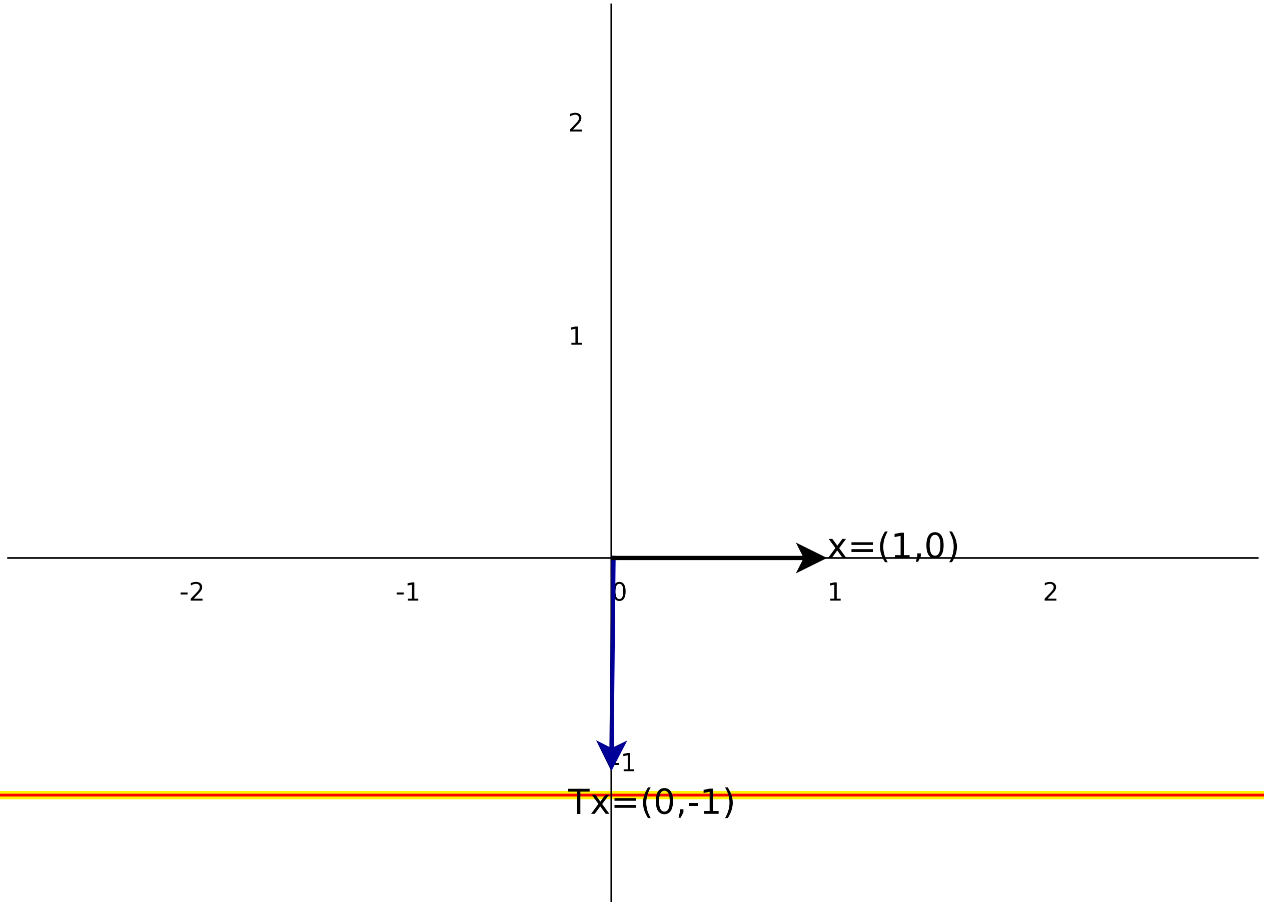

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