Rotations

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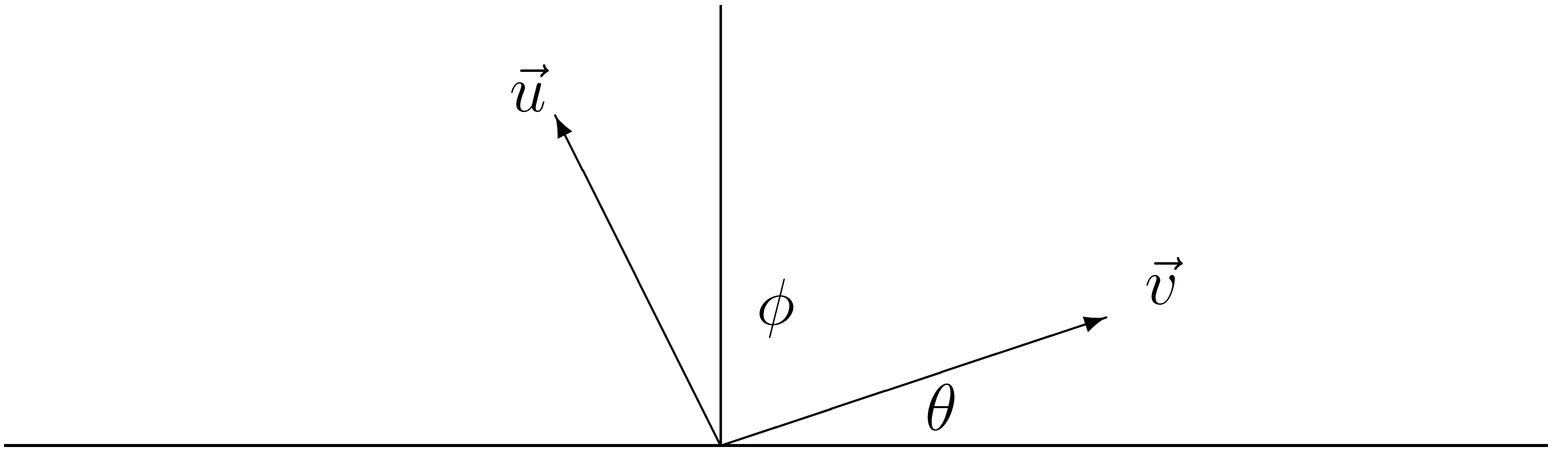
Rotations



Vector $\vec{v}$ has length r and
makes an angle θ with the x-axis.

Vector $\vec{u}$ has length r and
makes an angle $(\theta + \phi)$ with the x-axis.

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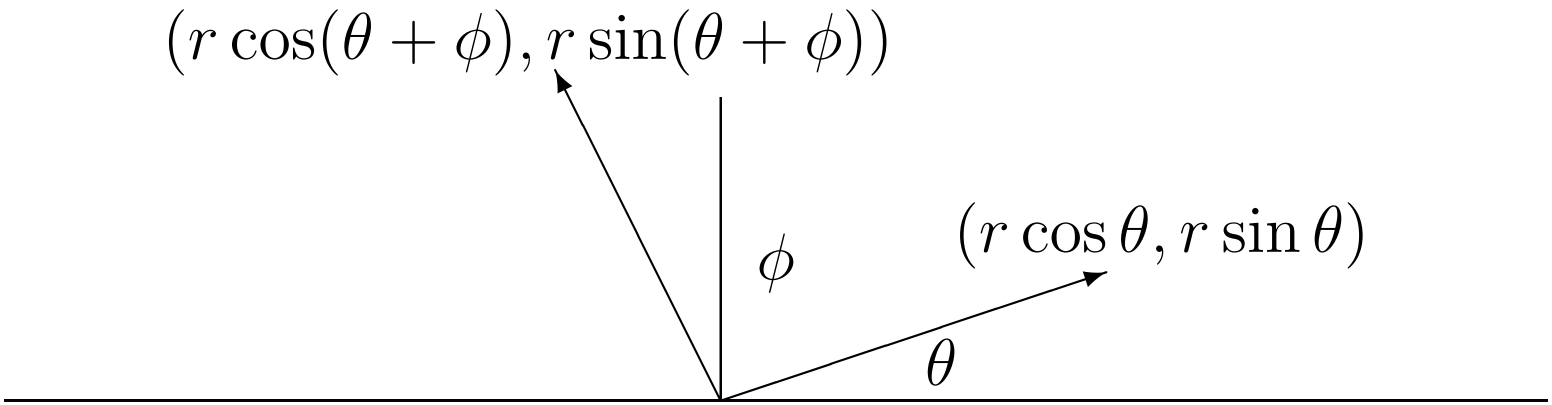

The polar coordinates of $\vec{v}$ are:

$$(r\cos\theta, r\sin\theta)$$

The polar coordinates of $\vec{u}$ are:

$$(r\cos(\theta+\phi), r\sin(\theta+\phi))$$

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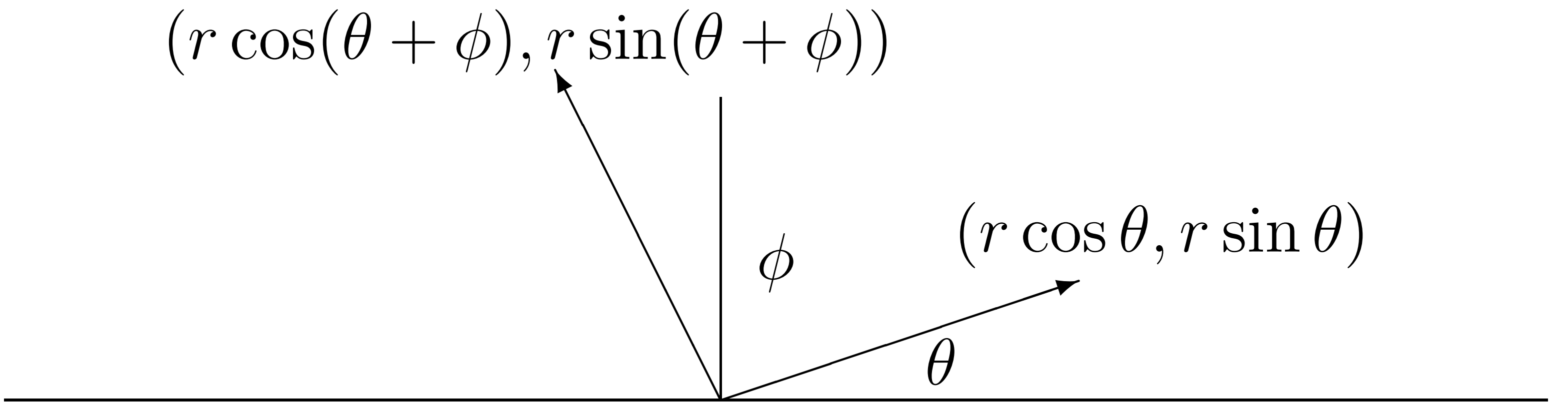

In vector form,

$$\vec{v} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \quad \vec{u} = \begin{bmatrix} r\cos(\theta+\phi) \\ r\sin(\theta+\phi) \end{bmatrix}$$

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Using the trigonometric identities

$$\begin{aligned}\cos(\theta+\phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi \\ \sin(\theta+\phi) &= \cos\theta\sin\phi + \sin\theta\cos\phi\end{aligned}$$

we can rewrite the vector $\vec{u}$ as

$$\vec{u} = \begin{bmatrix} r\cos(\theta+\phi) \\ r\sin(\theta+\phi) \end{bmatrix} = \begin{bmatrix} r\cos\theta\cos\phi - r\sin\theta\sin\phi \\ r\cos\theta\sin\phi + r\sin\theta\cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \vec{v} = A\vec{v}$$

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So a rotation through an angle θ is equivalent to multiplication by a matrix:

$$\vec{u} = \begin{bmatrix} r\cos(\theta+\phi) \\ r\sin(\theta+\phi) \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \vec{v} = A\vec{v}$$

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So a rotation through an angle θ is equivalent to multiplication by a matrix:

$$\vec{u} = \begin{bmatrix} r\cos(\theta+\phi) \\ r\sin(\theta+\phi) \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \vec{v} = A\vec{v}$$

This establishes that rotation is a kind of linear transformation,

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

defined by

$$T\vec{v} = A\vec{v} \quad \text{with} \quad A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

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Example: Consider the vector

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

rotated through an angle of $\phi = \pi/4$.

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$$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

rotated through an angle of $\phi = \pi/4$.

The matrix A associated with this transformation is

$$A = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$T\vec{v} = A\vec{v} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (\frac{1}{\sqrt{2}} - 0) \\ (\frac{1}{\sqrt{2}} + 0) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$