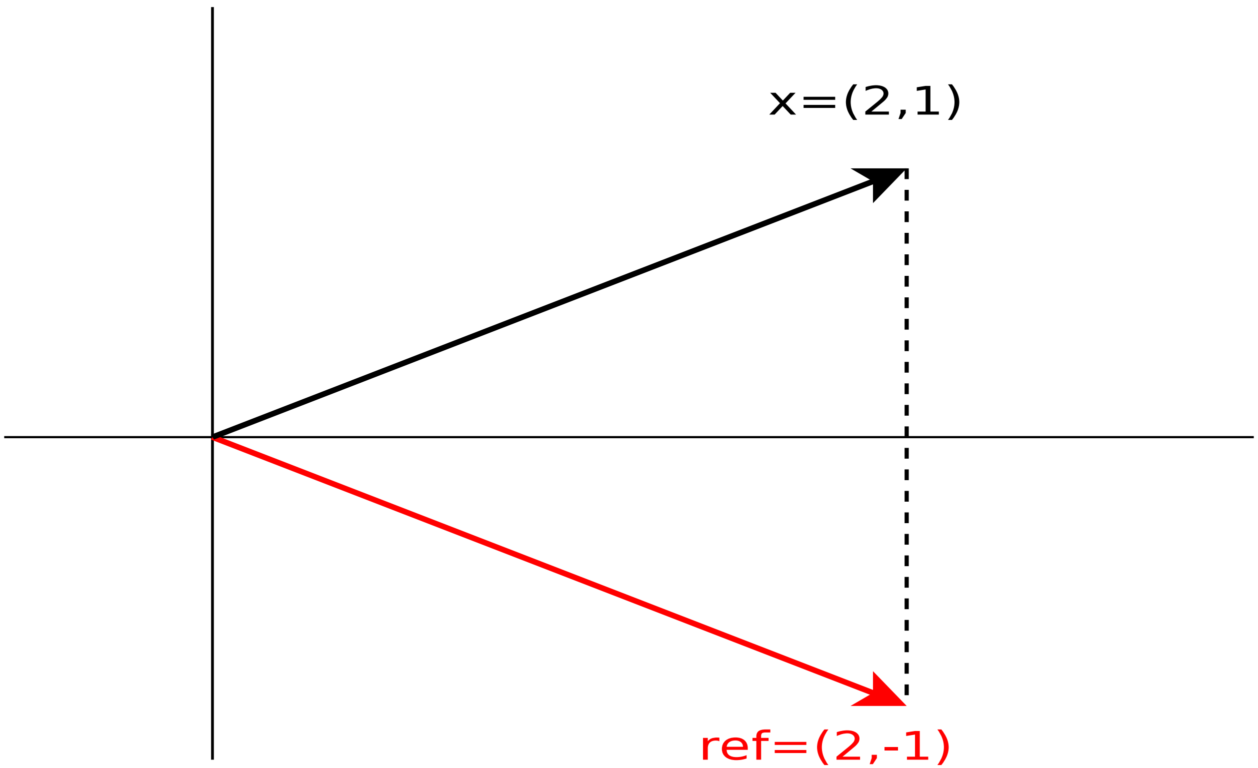
Reflections

Gene Quinn

Reflections

Suppose

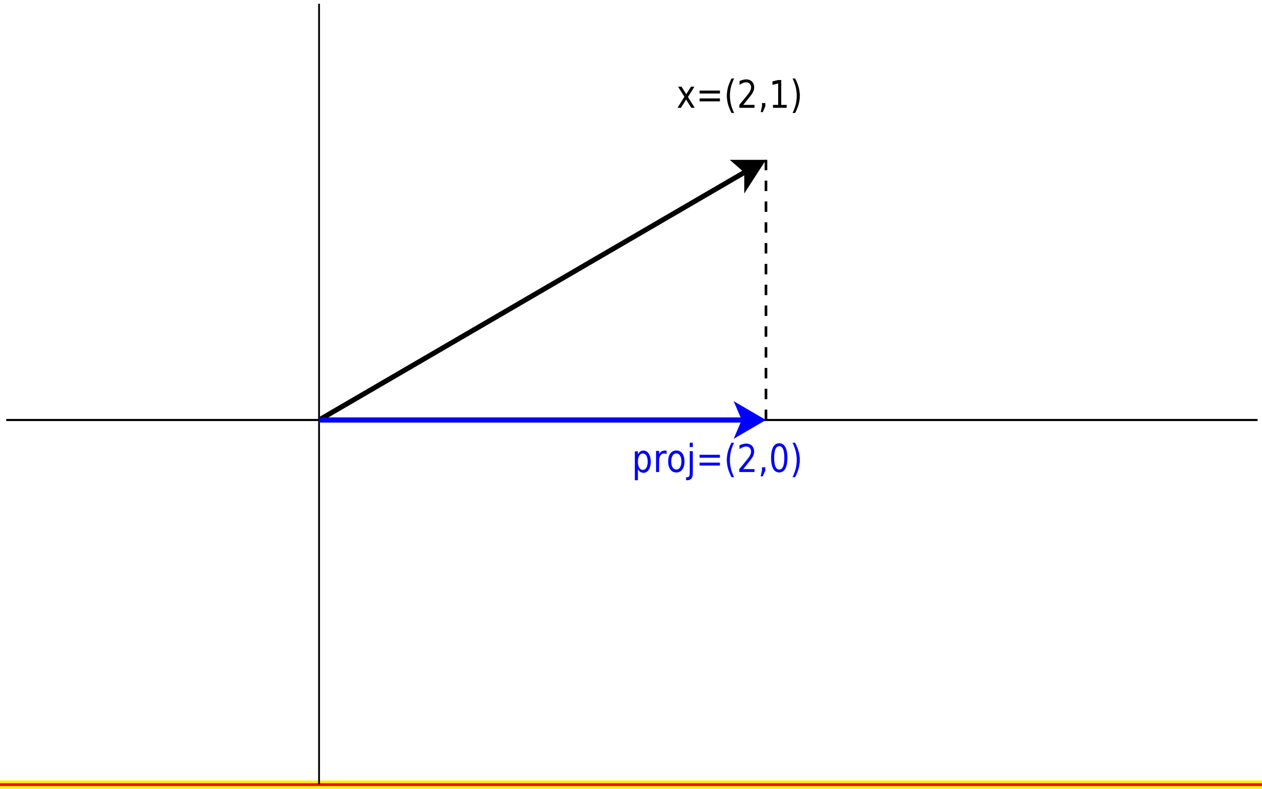
- $\vec{x} = (2, 1)$ is a vector
- ref$_X(\vec{x}) = (-2, 1)$ is $\vec{x}$ reflected across the x-axis



Reflections

Since $\vec{u} = (1, 0)$ is a unit vector parallel to the x-axis,

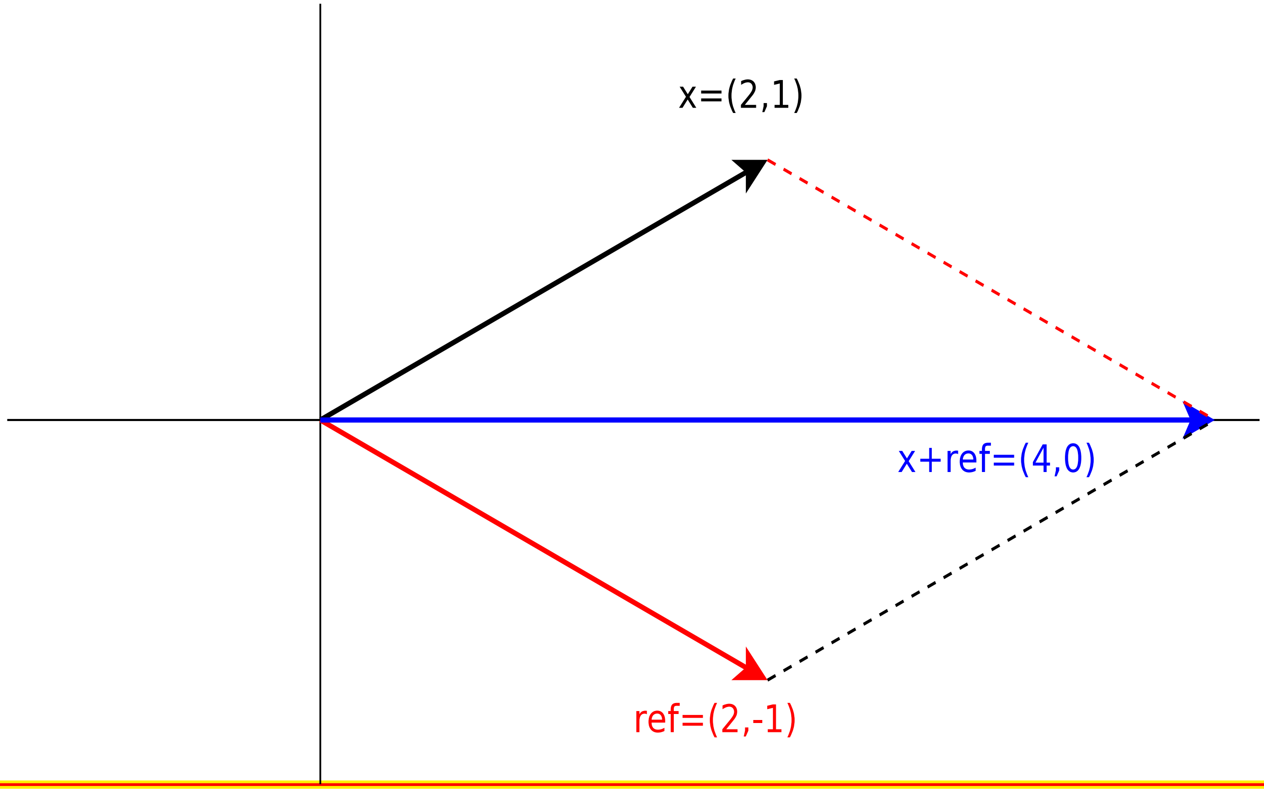
$$\text{proj}_X(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{u} = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$



Reflections

We can form the sum of $\vec{x}$ and its reflection $\text{ref}_X(\vec{x})$,

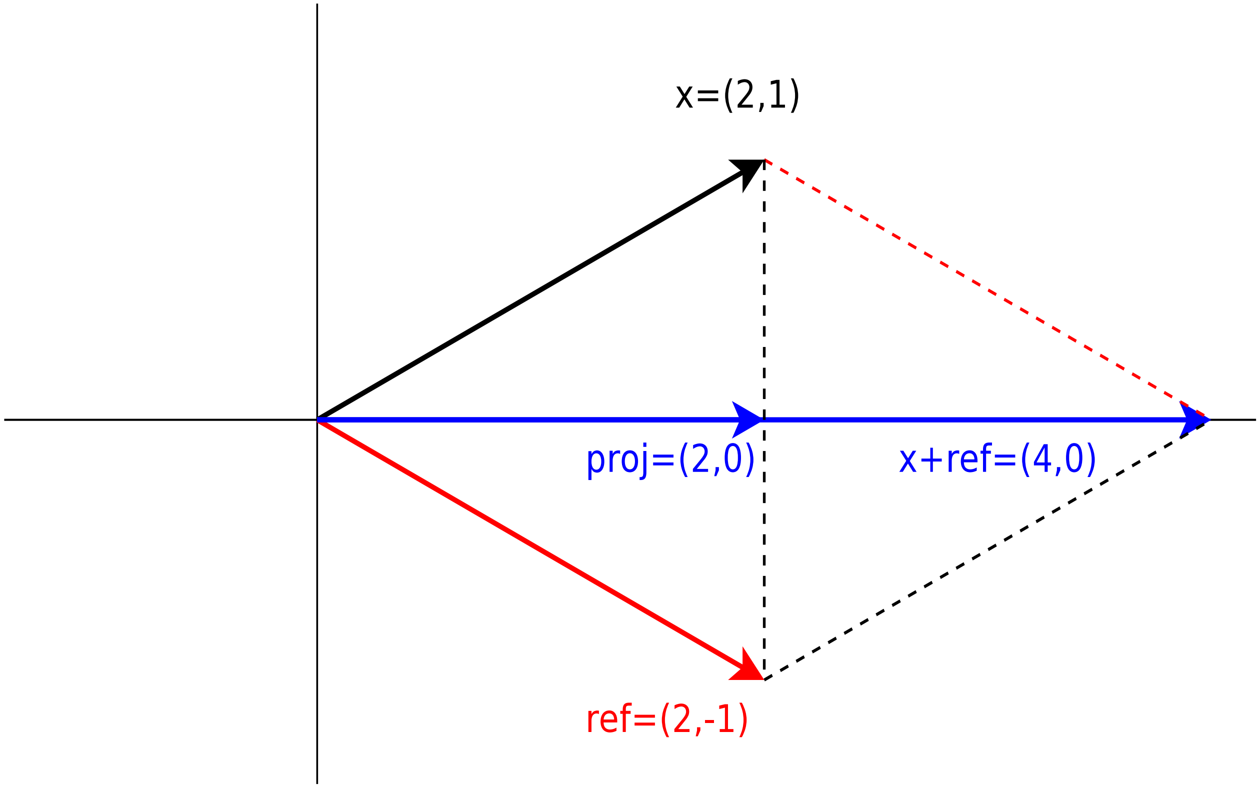
$$\vec{x} + \text{ref}_X(\vec{x}) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$



Reflections

Combining the two diagrams, we have:

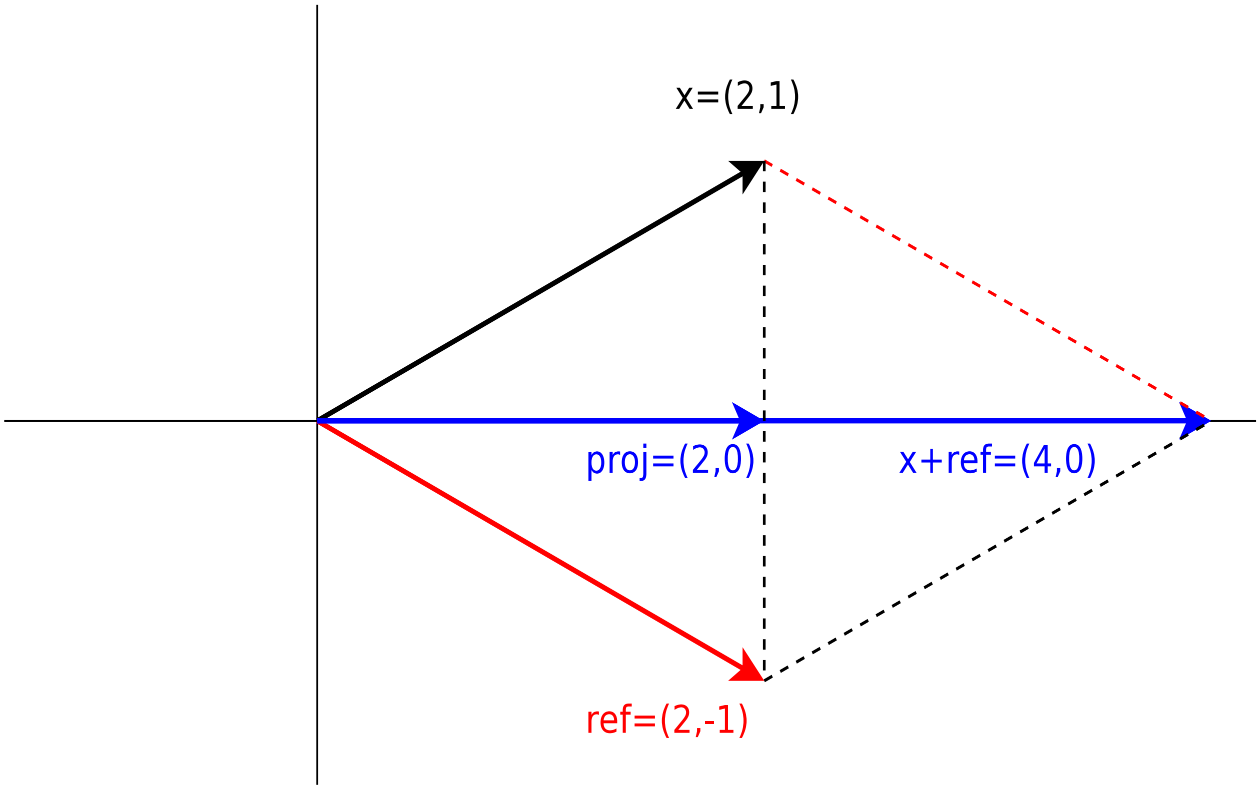
$$\vec{x} + \mathsf{ref}_X(\vec{x}) \;=\; 2\,\mathsf{proj}_X(\vec{x})$$



Reflections

Adding $-\vec{x}$ to both sides gives an expression for $\text{ref}_X(\vec{x})$:

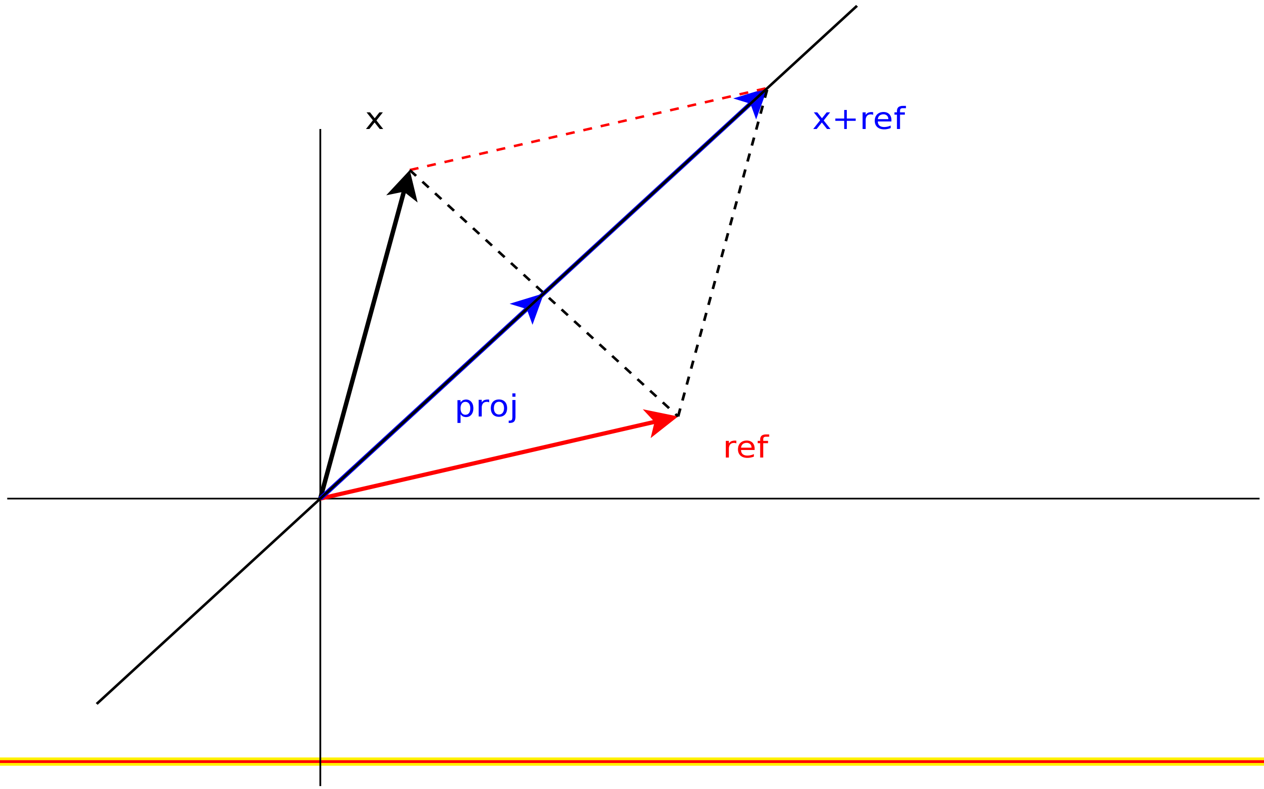
$$\text{ref}_X(\vec{x}) \;=\; 2\,\text{proj}_X(\vec{x}) \;-\; \vec{x}$$



Reflections

The fact from geometry that the diagonals of a parallelogram bisect each other means that our result will hold for more general reflections:

$$\vec{x} + \mathsf{ref}_X(\vec{x}) \;=\; 2\,\mathsf{proj}_X(\vec{x})$$



Reflections

Now let's examine the result algebraically:

$$\text{ref}_X(\vec{x}) \;=\; 2\,\text{proj}_X(\vec{x}) - \vec{x}$$

$$\text{ref}_X(\vec{x}) \;=\; 2\,(\vec{x}\cdot\vec{u})\vec{u}) - \vec{x}$$

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$$\text{ref}_X(\vec{x}) \;=\; 2\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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$$\text{ref}_X(\vec{x}) \;=\; 2\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Writing the dot product as $(x_1u_1 + x_2u_2)$ gives

$$\text{ref}_X(\vec{x}) \;=\; 2\,(x_1u_1 + x_2u_2)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Reflections

Carrying out the scalar multiplication, we get

$$\mathrm{ref}_X(\vec{x}) = \begin{bmatrix} 2u_1^2x_1 + 2u_1u_2x_2 \\ 2u_1u_2x_1 + 2u_2^2x_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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Carrying out the vector addition gives

$$\text{ref}_X(\vec{x}) \;=\; \begin{bmatrix} 2u_1^2x_1 - x_1 + 2u_1u_2x_2 \\ 2u_1u_2x_1 + 2u_2^2x_2 - x_2 \end{bmatrix}$$

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Finally, factoring out $\vec{x}$ yields

$$\text{ref}_X(\vec{x}) = \begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Reflections

So in two dimensions, reflection is a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with associated matrix

$$T\vec{x} \;=\; A\vec{x} \;=\; \mathsf{ref}_L(\vec{x})$$

with

$$A \;=\; \begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix}$$

Reflections

Here is a alternative (and simpler) way to determine the matrix A for reflections that makes use of our previous result for projections.

Start with our identity for $\text{ref}_L(\vec{x})$:

$$\text{ref}_L(\vec{x}) \;=\; 2\,\text{proj}_L(\vec{x}) - \vec{x}$$

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Start with our identity for ref$_L(\vec{x})$:

$$\text{ref}_L(\vec{x}) \;=\; 2\,\text{proj}_L(\vec{x}) - \vec{x}$$

Now make use of the fact that we already know the matrix for projections, call it A_p. We'll substitute

$$A_p\vec{x} \quad \text{for} \quad \text{proj}_L(\vec{x})$$

to obtain

$$\text{ref}_X(\vec{x}) \;=\; 2A_p\vec{x} \;-\; \vec{x}$$

Reflections

A special linear transformation known as the *identity transformation* maps a vector $\vec{x}$ into itself.

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A special linear transformation known as the *identity transformation* maps a vector $\vec{x}$ into itself.

Like every linear transformation, the identity transformation has an associated matrix A.

For $T_I : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, the matrix is

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Reflections

Using the definition of multiplication for matrices and vectors, we can verify the effect of the identity transform $T_I\vec{x} = \vec{x}$ on an arbitrary $\vec{x} \in \mathbb{R}^2$:

$$I\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \cdot x_1 + 0 \cdot x_2 \\ 0 \cdot x_1 + 1 \cdot x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{x}$$

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Now recall the expression for $\text{ref}_X(\vec{x})$:

$$\text{ref}_X(\vec{x}) = 2A_p\vec{x} - \vec{x}$$

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Now recall the expression for $\text{ref}_X(\vec{x})$:

$$\text{ref}_X(\vec{x}) = 2A_p\vec{x} - \vec{x}$$

We can substitute $I\vec{x}$ for $\vec{x}$ without changing the expression:

$$\text{ref}_X(\vec{x}) = 2A_p\vec{x} - I\vec{x}$$

Reflections

Now we can use the properties of matrix algebra to write the expression

$$\text{ref}_X(\vec{x}) \;=\; 2A_p\vec{x} \;-\; I\vec{x}$$

as

$$\text{ref}_X(\vec{x}) \;=\; (2A_p - I)\vec{x}$$

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This means that the matrix A_r associated with reflection is $2A_p - I$ where A_p is the matrix associated with projection.

Reflections

Using the properties of matrix addition, we can now determine A_r:

$$A_r = 2A_p - I = 2\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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$$A_r = \begin{bmatrix} 2u_1^2 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A_r = \begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix}$$

Reflections

So if $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a reflection, its associated matrix

$$T\vec{x} = A_r\vec{x}$$

is given by

$$A_r = \begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix}$$

where

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

is a unit vector parallel to the line of reflection L.

Reflections

Example: Find the matrix A_r of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ corresponding to reflection accross the line L with equation $y = x$.

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Example: Find the matrix A_r of the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ corresponding to reflection accross the line L with equation $y = x$.

The line L makes an angle of $\pi/4$ or 45 degrees with the x-axis, so the unit vector parallel to L is

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos \frac{\pi}{4} \\ \sin \frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Reflections

The matrix A_r is

$$A_r = \begin{bmatrix} 2u_1^2 - 1 & 2u_1 u_2 \\ 2u_1 u_2 & 2u_2^2 - 1 \end{bmatrix} = \begin{bmatrix} 2\left(\frac{1}{\sqrt{2}}\right)^2 - 1 & 2\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \\ 2\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} & 2\left(\frac{1}{\sqrt{2}}\right)^2 - 1 \end{bmatrix}$$

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so

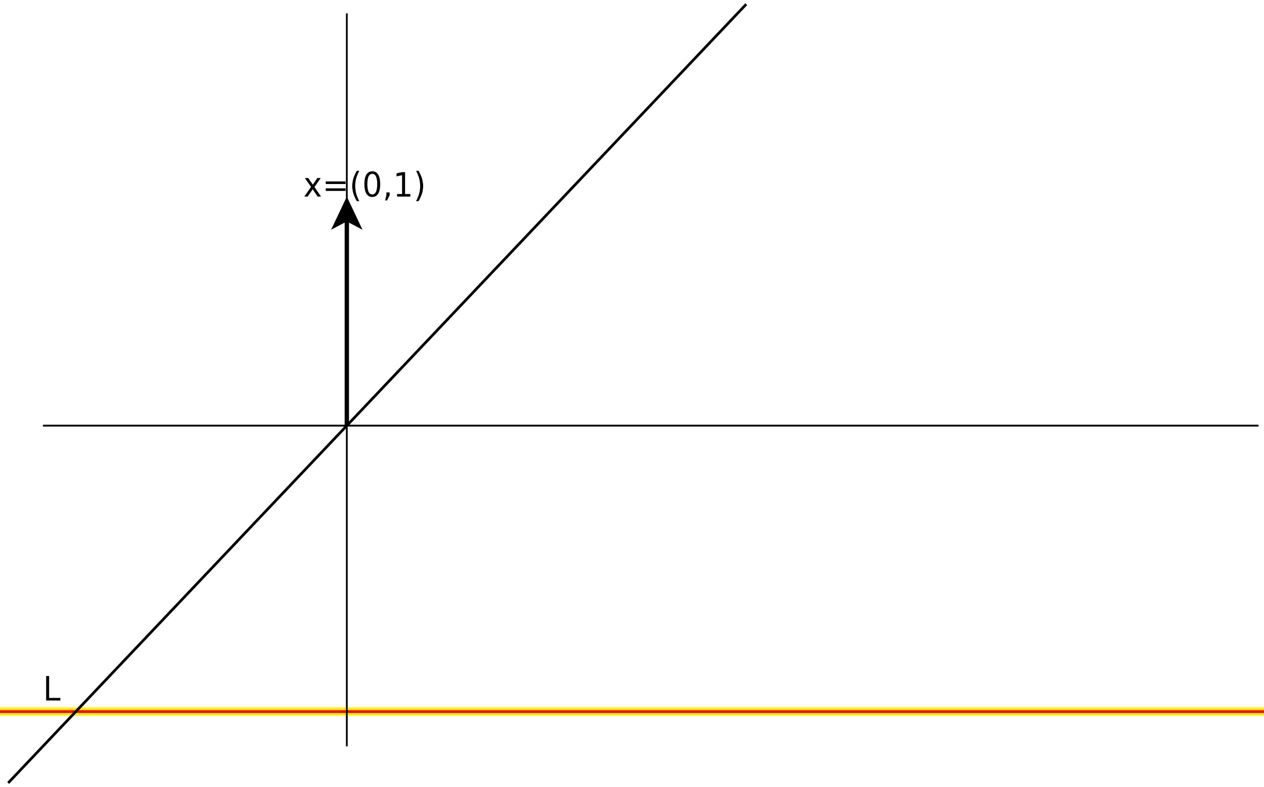
$$A_r = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Reflections

Example: Find the reflection of the vector

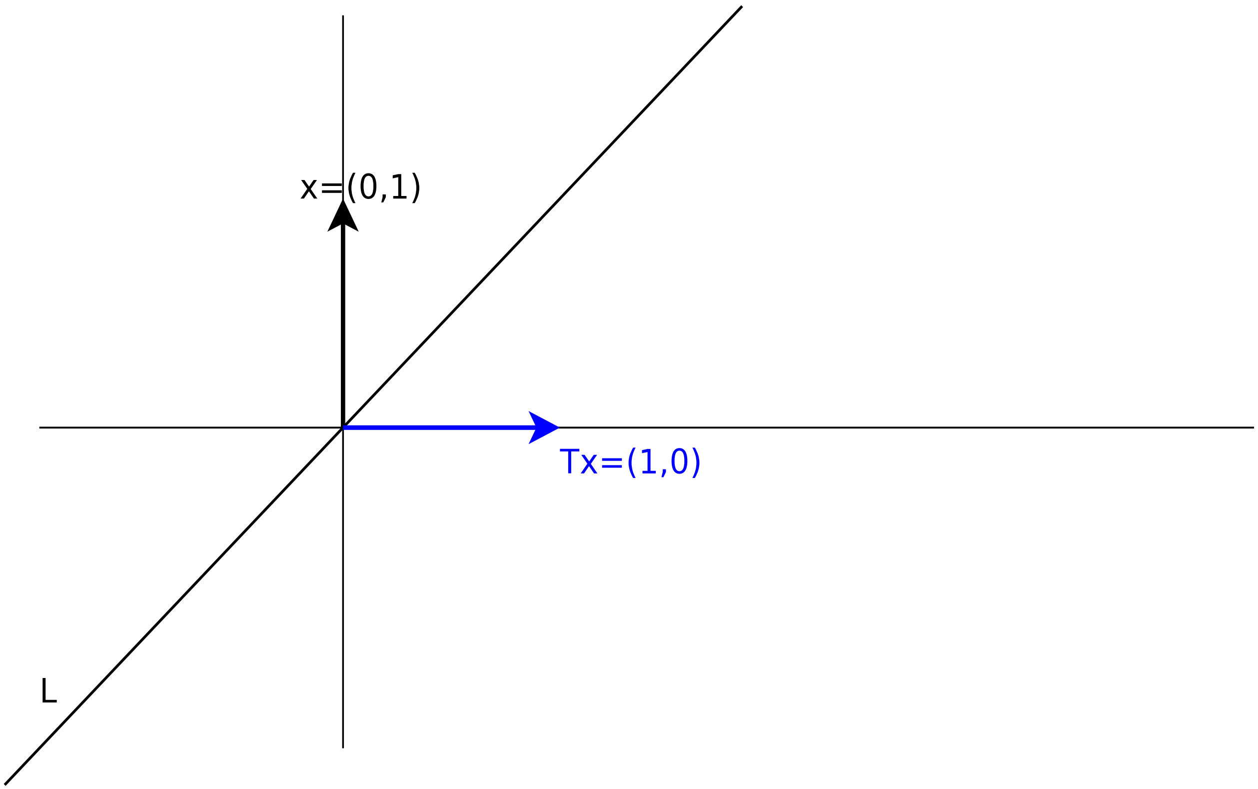
$$\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

across the line $y = x$:



Reflections

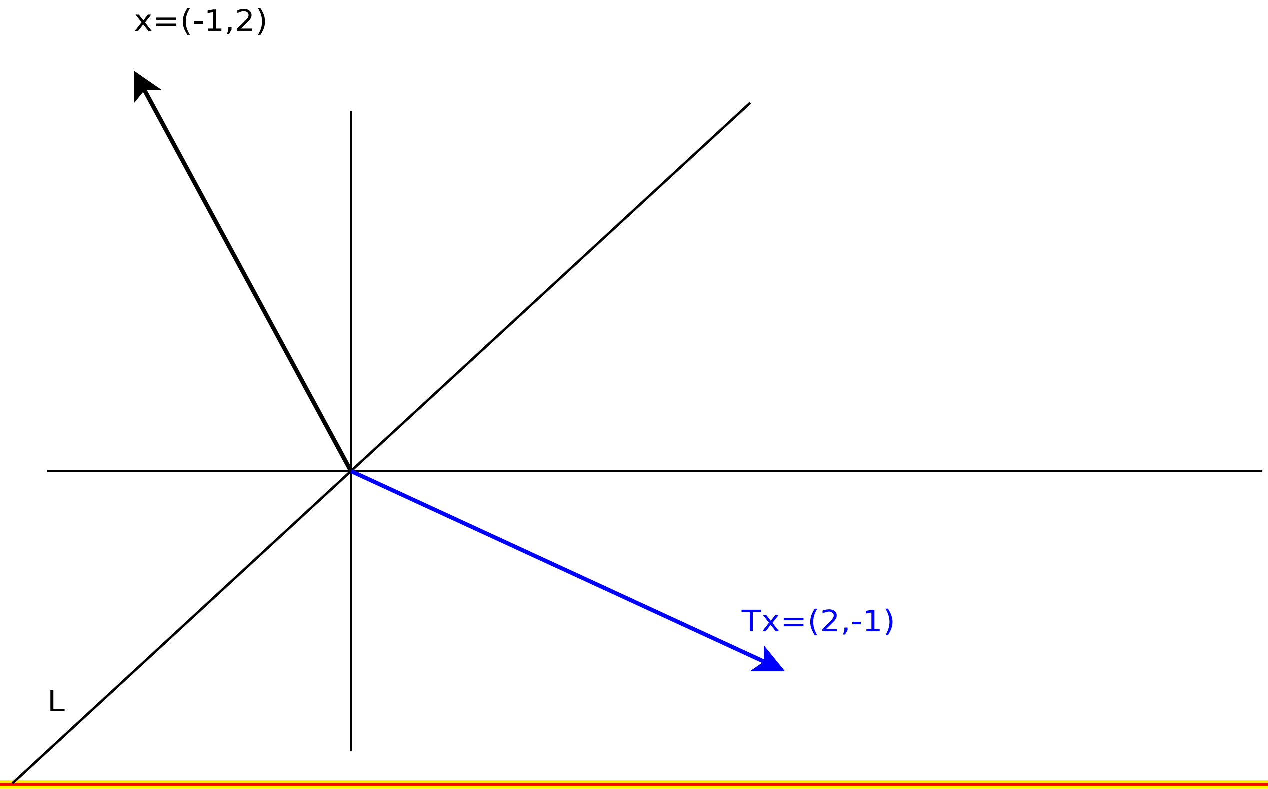
$$T\vec{x} = A\vec{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



Reflections

Find the reflection of $\vec{x} = (-1, 2)$ across the line $y = x$:

$$T\vec{x} = A\vec{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$



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Find the reflection of $\vec{x} = (1,2)$ across the line $y = x$:

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