Redundancy

Gene Quinn

Linear Independence and Redundancy

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In the text, linear independence is defined using the related concept of **redundancy**.

Redundancy

If

$$V = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$$

is a set of vectors $\vec{v}_i \in \mathbb{R}^n$, we say that the vector $\vec{v}_j$ is **redundant** if $\vec{v}_j$ is a linear combination of the vectors with $\vec{v}_1, \ldots, \vec{v}_{j-1}$.

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That is, if there exists scalars $c_1, \ldots, c_{j-1}$ with the property that

$$\vec{v}_j = c_1\vec{v}_1 + \cdots + c_{j-1}\vec{v}_{j-1}$$

Redundancy

Example: Consider the following set of vectors:

$$V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$$

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$\vec{v}_3$ is redundant because it is a linear combination of $\vec{v}_1$ and $\vec{v}_2$:

$$\vec{v}_1 + 2\vec{v}_2 = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \vec{v}_3$$

Redundancy

Example: It's not always easy to tell if vectors are redundant. Suppose

$$V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ -4 \end{bmatrix} \right\}$$

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$\vec{v}_3$ is redundant because it is a linear combination of $\vec{v}_1$ and $\vec{v}_2$:

$$2\vec{v}_1 - 3\vec{v}_2 = 2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 4 \end{bmatrix} = \vec{v}_3$$

Redundancy

In general, how do we determine whether a set has any redundant vectors?

Here is a procedure that always works:

First, form a matrix A whose columns are the vectors in the set V:

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \end{bmatrix}$$

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First, form a matrix A whose columns are the vectors in the set V:

$$A = \left[\begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \end{array} \right]$$

Next, compute $rref(A)$.

If every one of the m columns of $rref(A)$ contains a leading 1, there are no redundant vectors in V.

Redundancy

Example: Suppose as before

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Form a matrix A whose columns are the elements of V:

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 10 \\ 1 & -1 & 5 \\ 1 & 2 & -4 \end{bmatrix}$$

Redundancy

Compute the reduced row-echelon form $rref(A)$:

$$rref(A) = rref\begin{bmatrix} 2 & -2 & 10 \\ 1 & -1 & 5 \\ 1 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

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If we associate the first column of A and $rref(A)$ with $\vec{v}_1$, the second with $\vec{v}_2$, and so on, the vector corresponding to the first column without a leading 1 is redundant.

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In this case, $\vec{v}_3$ is redundant because the first column without a leading 1 is column 3, which corresponds to $\vec{v}_3$.

We can also tell from $rref(A)$ that $2\vec{v}_1 - 3\vec{V}_2 = \vec{v}_3$.

Redundancy

Example: This time let

$$V = \{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} 3 \\ 1 \\ 5 \\ 6 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 2 \\ 3 \end{bmatrix} \right\}$$

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Form a matrix A whose columns are the elements of V:

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 4 \\ 5 & 2 \\ 6 & 3 \end{bmatrix}$$

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Based on Definition 3.2.3 on page 115, we would also say that $\vec{v}_1$ and $\vec{v}_2$ are *linearly independent*.

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Example: This time suppose

$$V = \left\{ \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} -2 \\ 14 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} \right\}$$

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Form a matrix A whose columns are the elements of V:

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -2 & 2 & -1 \\ 1 & 4 & 14 & 5 & 3 \\ 3 & 2 & 2 & 7 & 5 \end{bmatrix}$$

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From $rref(A)$ we see that $\vec{v}_3$ and $\vec{v}_5$ are redundant.

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In general, the subset of V consisting of all of the elements of V corresponding to columns of $rref(A)$ that contain leading ones is a linearly independent set of vectors.