Proof that $ker(T)$ is a Subspace of $\mathbb{R}^n$

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Kernels and Subspaces

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is a linear transformation with domain $\mathbb{R}^m$ and codomain $\mathbb{R}^n$.

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In this situation, the *kernel* of T is defined as:

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The set-builder notation would be read as "The set of all vectors $\vec{x}$ in $\mathbb{R}^m$ such that $T(\vec{x})$ is the zero vector in $\mathbb{R}^n$ ".

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Theorem: For any linear transformation

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Proof: In order to prove that $ker(T)$ is a subspace, we must establish the following three claims:

- $\vec{0}_m \in ker(T)$
- $ker(T)$ is closed under addition
- $ker(T)$ is closed under scalar multiplication

Kernels and Subspaces

Claim 1: $\vec{0}_m \in ker(T)$

Proof of Claim 1: By an earlier theorem, there exists an $n \times m$ matrix A with the property that

$$T(\vec{x}) \;=\; A\vec{x} \quad \forall \vec{x} \in \mathbb{R}^m$$

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Let $\vec{x}$ be the zero vector in $\mathbb{R}^m$, $\vec{0}_m$.

Then by the properties of matrix multiplication,

$$A\vec{0}_m \;=\; \vec{0}_n$$

for any $n \times m$ matrix A. Therefore, $A\vec{0}_m = T(\vec{0}_m) = \vec{0}_n$, and so by definition $\vec{0}_m$ is in $ker(T)$.

Kernels and Subspaces

Claim 2: $ker(T)$ is closed under addition.

Proof of Claim 2: Let $\vec{u}, \vec{v} \in \mathbb{R}^m$ be arbitrary elements of $ker(T)$. We need to show that $\vec{u} + \vec{v} \in ker(T)$.

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By the definition of $ker(T)$,

$$T(\vec{u}) = A\vec{u} = \vec{0}_n \quad \text{and} \quad T(\vec{v}) = A\vec{v} = \vec{0}_n$$

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By the properties of linear transformations,

$$T(\vec{u} + \vec{v}) = A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \vec{0}_n + \vec{0}_n = \vec{0}_n$$

and therefore $\vec{u} + \vec{v} \in ker(T)$.

Kernels and Subspaces

Claim 3: $ker(T)$ is closed under scalar multiplication.

Proof of Claim 3: Let $\vec{u} \in \mathbb{R}^n$ be an arbitrary element of $ker(T)$ and $k \in \mathbb{R}$ an arbitrary scalar. We need to show that $k\vec{u} \in ker(T)$.

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By the definition of $ker(T)$,

$$T(\vec{u}) = A\vec{u} = \vec{0}_n$$

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This completes the proof that $ker(T)$ is a subspace.