Proof that $im(\mathbb{R}^m)$ is a Subspace of $\mathbb{R}^n$

Gene Quinn

Images and Subspaces

Suppose

$$T : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

is a linear transformation with domain $\mathbb{R}^m$ and codomain $\mathbb{R}^n$.

Images and Subspaces

Suppose

$$T : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

is a linear transformation with domain $\mathbb{R}^m$ and codomain $\mathbb{R}^n$.

In this situation, the *image* under T of $\mathbb{R}^m$ is defined as:

$$im(T) \ = \ \{\vec{y} \in \mathbb{R}^n \ : \ \exists \vec{x} \in \mathbb{R}^m \text{ such that } T(\vec{x}) = \vec{y}\}$$

Images and Subspaces

Suppose

$$T : \mathbb{R}^m \rightarrow \mathbb{R}^n$$

is a linear transformation with domain $\mathbb{R}^m$ and codomain $\mathbb{R}^n$.

In this situation, the *image* under T of $\mathbb{R}^m$ is defined as:

$$im(T) \ = \ \{\vec{y} \in \mathbb{R}^n \ : \ \exists \vec{x} \in \mathbb{R}^m \text{ such that } T(\vec{x}) = \vec{y}\}$$

The set-builder notation would be read as "The set of all vectors $\vec{y}$ in $\mathbb{R}^n$ for which there exists a vector $\vec{x}$ in $\mathbb{R}^m$ such that $T(\vec{x}) = \vec{y}$ ".

Images and Subspaces

Theorem: For any linear transformation

$$T : \mathbb{R}^m \to \mathbb{R}^n,$$

the image under T of $\mathbb{R}^m$, $im(T)$, is a subspace (of $\mathbb{R}^n$).

Images and Subspaces

Theorem: For any linear transformation

$$T : \mathbb{R}^m \to \mathbb{R}^n,$$

the image under T of $\mathbb{R}^m$, $im(T)$, is a subspace (of $\mathbb{R}^n$).

Proof: In order to prove that $im(T)$ is a subspace, we must establish the following three claims:

- $\vec{0}_n \in im(T)$
- $im(T)$ is closed under addition
- $im(T)$ is closed under scalar multiplication

Images and Subspaces

Claim 1: $\vec{0}_n \in im(T)$

Proof of Claim 1: By an earlier theorem, there exists an $n \times m$ matrix A with the property that

$$T(\vec{x}) \;=\; A\vec{x} \quad \forall \vec{x} \in \mathbb{R}^m$$

Images and Subspaces

Claim 1: $\vec{0}_n \in im(T)$

Proof of Claim 1: By an earlier theorem, there exists an $n \times m$ matrix A with the property that

$$T(\vec{x}) \;=\; A\vec{x} \quad \forall \vec{x} \in \mathbb{R}^m$$

Let $\vec{x}$ be the zero vector in $\mathbb{R}^m$, $\vec{0}_m$.

Then by the properties of matrix multiplication,

$$A\vec{0}_m \;=\; \vec{0}_n$$

for any $n \times m$ matrix A. Therefore, $A\vec{0}_m = T(\vec{0}_m) = \vec{0}_n$, and so by definition $\vec{0}_n$ is in $im(T)$.

Images and Subspaces

Claim 2: $im(T)$ is closed under addition.

Proof of Claim 2: Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ be arbitrary elements of $im(T)$. We need to show that $\vec{u} + \vec{v} \in im(T)$.

Images and Subspaces

Claim 2: $im(T)$ is closed under addition.

Proof of Claim 2: Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ be arbitrary elements of $im(T)$. We need to show that $\vec{u} + \vec{v} \in im(T)$.

By the definition of $im(T)$, there are vectors $\vec{x}, \vec{y} \in \mathbb{R}^m$ such that

$$T(\vec{x}) = A\vec{x} = \vec{u} \quad \text{and} \quad T(\vec{y}) = A\vec{y} = \vec{v}$$

Images and Subspaces

Claim 2: $im(T)$ is closed under addition.

Proof of Claim 2: Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ be arbitrary elements of $im(T)$. We need to show that $\vec{u} + \vec{v} \in im(T)$.

By the definition of $im(T)$, there are vectors $\vec{x}, \vec{y} \in \mathbb{R}^m$ such that

$$T(\vec{x}) = A\vec{x} = \vec{u} \quad \text{and} \quad T(\vec{y}) = A\vec{y} = \vec{v}$$

Since $\mathbb{R}^m$ is a subspace of itself, it is closed under addition, and therefore $\vec{x} + \vec{y} \in \mathbb{R}^m$.

Images and Subspaces

Claim 2: $im(T)$ is closed under addition.

Proof of Claim 2: Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ be arbitrary elements of $im(T)$. We need to show that $\vec{u} + \vec{v} \in im(T)$.

By the definition of $im(T)$, there are vectors $\vec{x}, \vec{y} \in \mathbb{R}^m$ such that

$$T(\vec{x}) = A\vec{x} = \vec{u} \quad \text{and} \quad T(\vec{y}) = A\vec{y} = \vec{v}$$

Since $\mathbb{R}^m$ is a subspace of itself, it is closed under addition, and therefore $\vec{x} + \vec{y} \in \mathbb{R}^m$.

By the properties of linear transformations,

$$T(\vec{x} + \vec{y}) \; = \; A(\vec{x} + \vec{y}) \; = \; A\vec{x} + A\vec{y} \; = \; \vec{u} + \vec{v}$$

and therefore $\vec{u} + \vec{v} \in im(T)$.

Images and Subspaces

Claim 3: $im(T)$ is closed under scalar multiplication.

Proof of Claim 3: Let $\vec{u} \in \mathbb{R}^n$ be an arbitrary element of $im(T)$ and $k \in \mathbb{R}$ an arbitrary scalar. We need to show that $k\vec{u} \in im(T)$.

Images and Subspaces

Claim 3: $im(T)$ is closed under scalar multiplication.

Proof of Claim 3: Let $\vec{u} \in \mathbb{R}^n$ be an arbitrary element of $im(T)$ and $k \in \mathbb{R}$ an arbitrary scalar. We need to show that $k\vec{u} \in im(T)$.

By the definition of $im(T)$, there is a vector $\vec{x} \in \mathbb{R}^m$ such that

$$T(\vec{x}) = A\vec{x} = \vec{u}$$

Images and Subspaces

Since $\mathbb{R}^m$ is a subspace of itself, it is closed under scalar multiplication, and therefore for any real number k, $k\vec{x} \in \mathbb{R}^m$.

Images and Subspaces

Since $\mathbb{R}^m$ is a subspace of itself, it is closed under scalar multiplication, and therefore for any real number k, $k\vec{x} \in \mathbb{R}^m$.

By the properties of linear transformations,

$$T(k\vec{x}) = A(k\vec{x}) = kA\vec{x} = k\vec{u}$$

and therefore $k\vec{u} \in im(T)$.

Images and Subspaces

Since $\mathbb{R}^m$ is a subspace of itself, it is closed under scalar multiplication, and therefore for any real number k, $k\vec{x} \in \mathbb{R}^m$.

By the properties of linear transformations,

$$T(k\vec{x}) = A(k\vec{x}) = kA\vec{x} = k\vec{u}$$

and therefore $k\vec{u} \in im(T)$.

This completes the proof that $im(T)$ is a subspace.