Proof that $im(A)$ is the Span of the Columns of A

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$im(A)$ and span

If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation with associated $n \times m$ matrix A, and we write the columns of A

$$A = [\vec{a}_1 \ \cdots \vec{a}_m]$$

as a set of m column vectors,

$$V = \{\vec{a}_1, \ldots, \vec{a}_m\}$$

then

$$im(A) = \text{span}(V)$$

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Proof: Since $im(A)$ and span(V) are sets, the proof requires us to show that two sets are the same.

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To prove that $im(A) \subseteq$ span(V), first assume that a vector $\vec{x}$ belongs to span(V).

If we can show that this implies that it also belongs to $im(A)$, then we have established that span$(V) \subseteq im(A)$.

$im(A)$ and span

First suppose $\vec{x} \in \text{span}(V)$.

Then for some scalars $c_1, \ldots, c_m$,

$$\vec{x} \;=\; c_1\vec{a}_1 + \cdots + c_m\vec{a}_m$$

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But this is the same as

$$\vec{x} = [\vec{a}_1 \cdots \vec{a}_m] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} = A\vec{c}$$

where $\vec{c} = (c_1, \ldots, c_m) \in \mathbb{R}^m$, so $\vec{x} \in im(A)$.

$im(A)$ and span

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But this is the same as

$$\vec{x} \;=\; [\vec{a}_1 \cdots \vec{a}_m] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} \;=\; c_1\vec{a}_1 + \cdots + c_n\vec{a}_m$$

so $x \in$ span $(\{\vec{a} \cdots \vec{a}_m\}) \;=\;$ span(V).

$im(A)$ and span

By establishing that

$$im(A) \subseteq \text{span}(V) \quad \text{and} \quad \text{span}(V) \subseteq im(A)$$

we have shown that

$$im(A) = \text{span}(V)$$