Projections

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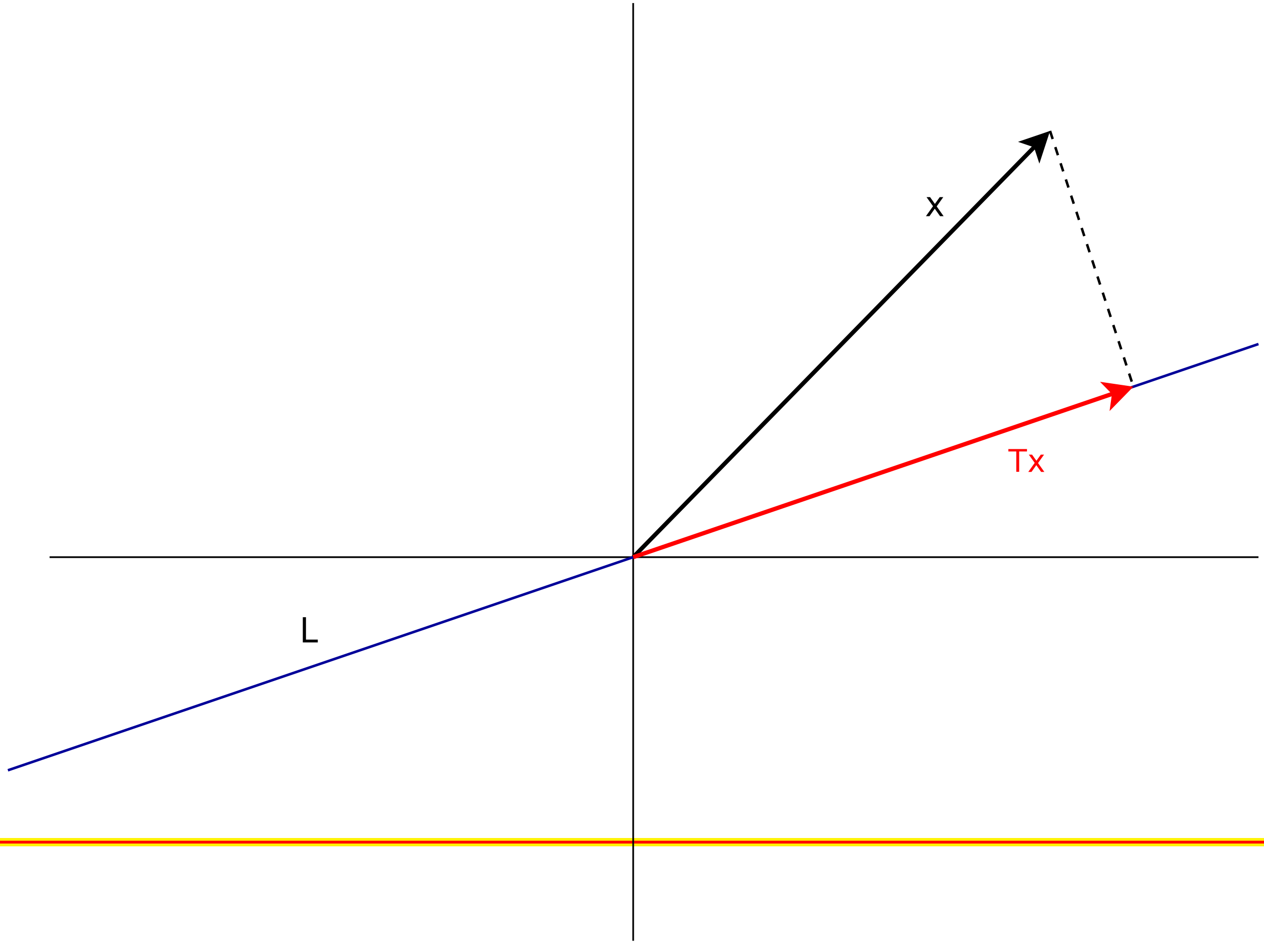
Projections

A **projection** is a type of linear transformation

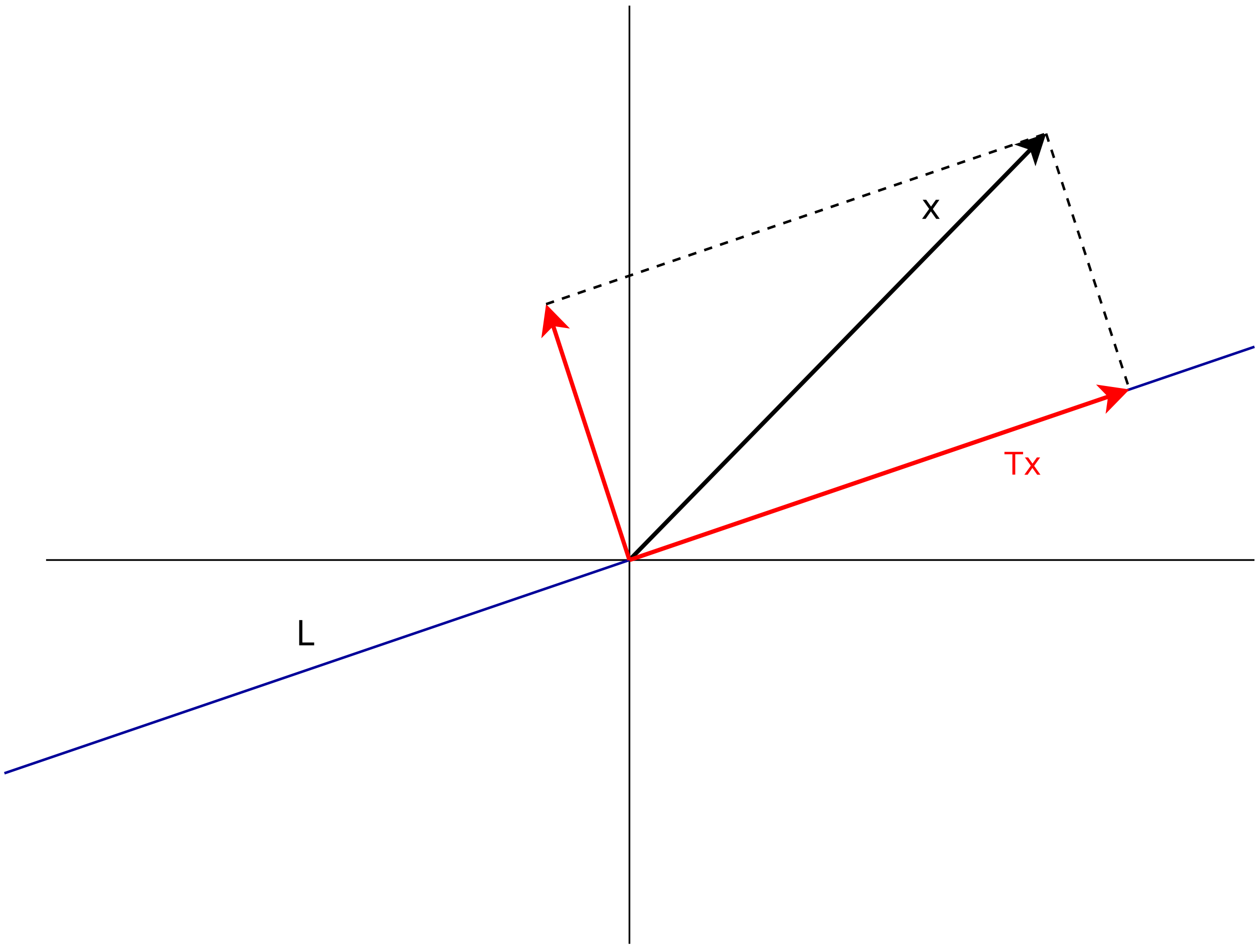
$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

that *projects* a vector $\vec{x}$ onto a line L.



Projections

The key idea is to write $\vec{x}$ as the sum of a vector parallel to L, $\vec{x}^{\parallel}$ and a vector $\vec{x}^{\perp}$ perpendicular to L:



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The first step is to obtain a unit vector $\vec{u}$ in the direction of the line L.

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Suppose $\vec{l}$ is a vector parallel to the line L. Then

$$\vec{u} = \frac{\vec{l}}{\|\vec{l}\|} = \left(\frac{1}{\sqrt{\vec{l}\cdot\vec{l}}}\right)\vec{l}$$

is a unit vector in the direction of L.

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is a unit vector in the direction of L.

The projection of $\vec{x}$ onto line L is then given by

$$\text{proj}_L(\vec{x}) = (\vec{x}\cdot\vec{u})\,\vec{u}$$

Projections

Expanding $(\vec{x} \cdot \vec{u})\vec{u}$ into its components,

$$(\vec{x} \cdot \vec{u})\,\vec{u} = \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

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Expanding the dot product $\vec{x} \cdot \vec{u}$ into $x_1 u_1 + x_2 v_2$, this becomes

$$(x_1 u_1 + x_2 u_2) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

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$$= \begin{bmatrix} u_1^2 x_1 + u_1 u_2 x_2 \\ u_1 u_2 x_1 + u_2^2 x_2 \end{bmatrix} = \begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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Now we can write projection onto L for any vector $\vec{x}$ as a linear transformation with the following matrix A:

$$T\vec{x} \;=\; A\vec{x} \;=\; \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix} \vec{x}$$

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In the above formula, u_1 and u_2 are the components of a unit vector $\vec{u}$ which is parallel to the line L.

Projections

Suppose we want to find the matrix A that projects a vector $\vec{x}$ onto the line $y = x$ (i.e., the line that passes through the origin at a 45 degree angle).

The unit vector $\vec{u}$ along this line is

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

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Now the matrix A is

$$A = \begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Projections

Example: Find the projection of the vector $\vec{x} = (1, 2)$ onto the line L that passes through the origin at a 45 degree angle.

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We determined the matrix A for this projection on the previous slide. The projection of $\vec{x} = (1, 2)$ is

$$T\vec{x} \;=\; A\vec{x} \;=\; \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

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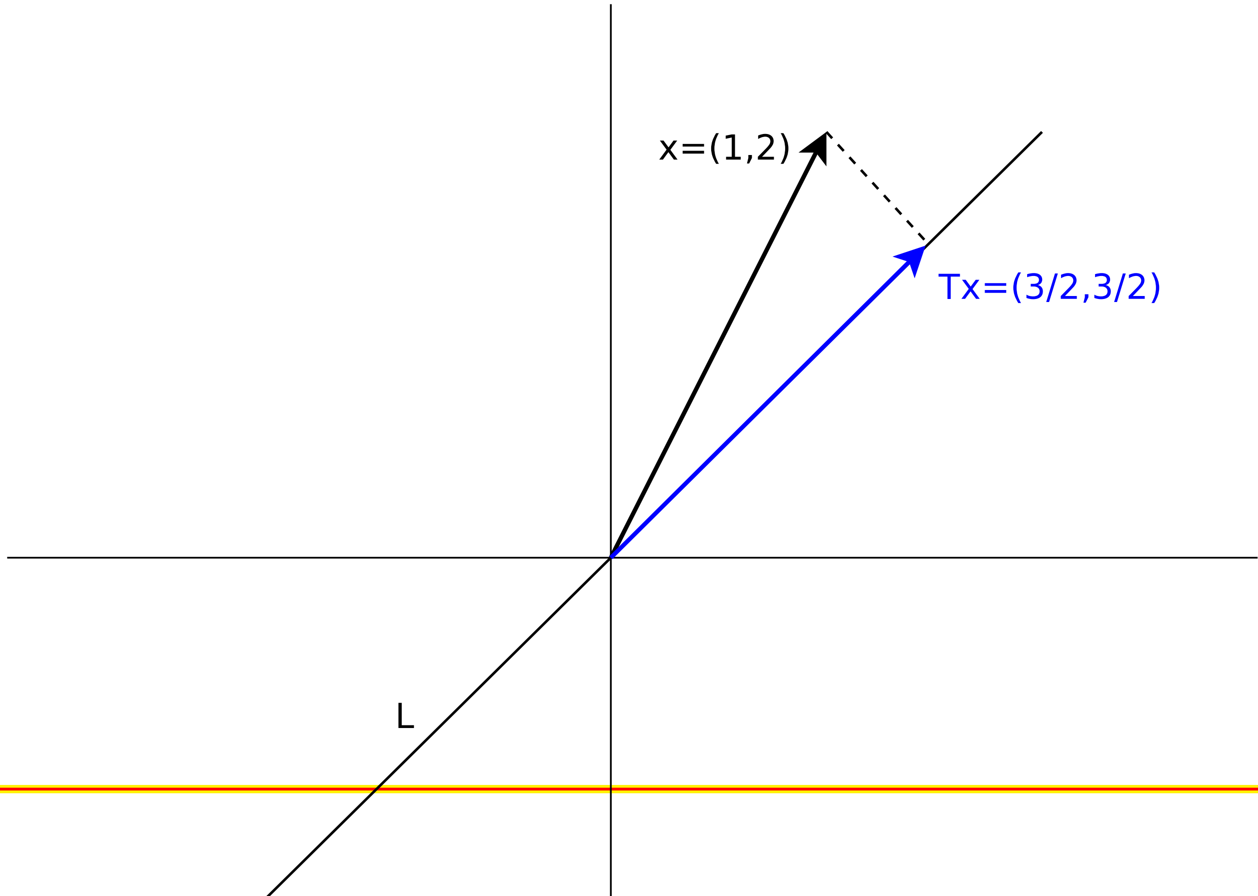
The projection $A\vec{x} = \vec{x}^{\|}$ is

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{2}{2} \\ \frac{1}{2} + \frac{2}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix} = \vec{x}^{\|}$$

Projections

The projection is

$$T\vec{x} = A\vec{x} = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix} = \vec{x}^{\parallel}$$



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Example: Find the matrix A with the property that, for any vector $\vec{x}$,

$$T\vec{x} = A\vec{x}$$

is the projection of $\vec{x}$ onto the line through the origin that makes an angle of $\pi/6$ or 30 degrees with the horizontal axis.

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If we have an angle θ with the horizontal axis, the unit vector in that direction is

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

Projections

In terms of θ, the projection matrix A is:

$$A = \begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

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If $\theta = \pi/6$, then

$$\cos\theta = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin\theta = \frac{1}{2}$$

Projections

The matrix A of the projection onto this line is then

$$A = \begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}$$

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$$A = \begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}$$

Now for any matrix $\vec{w}$, the projection of $\vec{w}$ onto the line L is

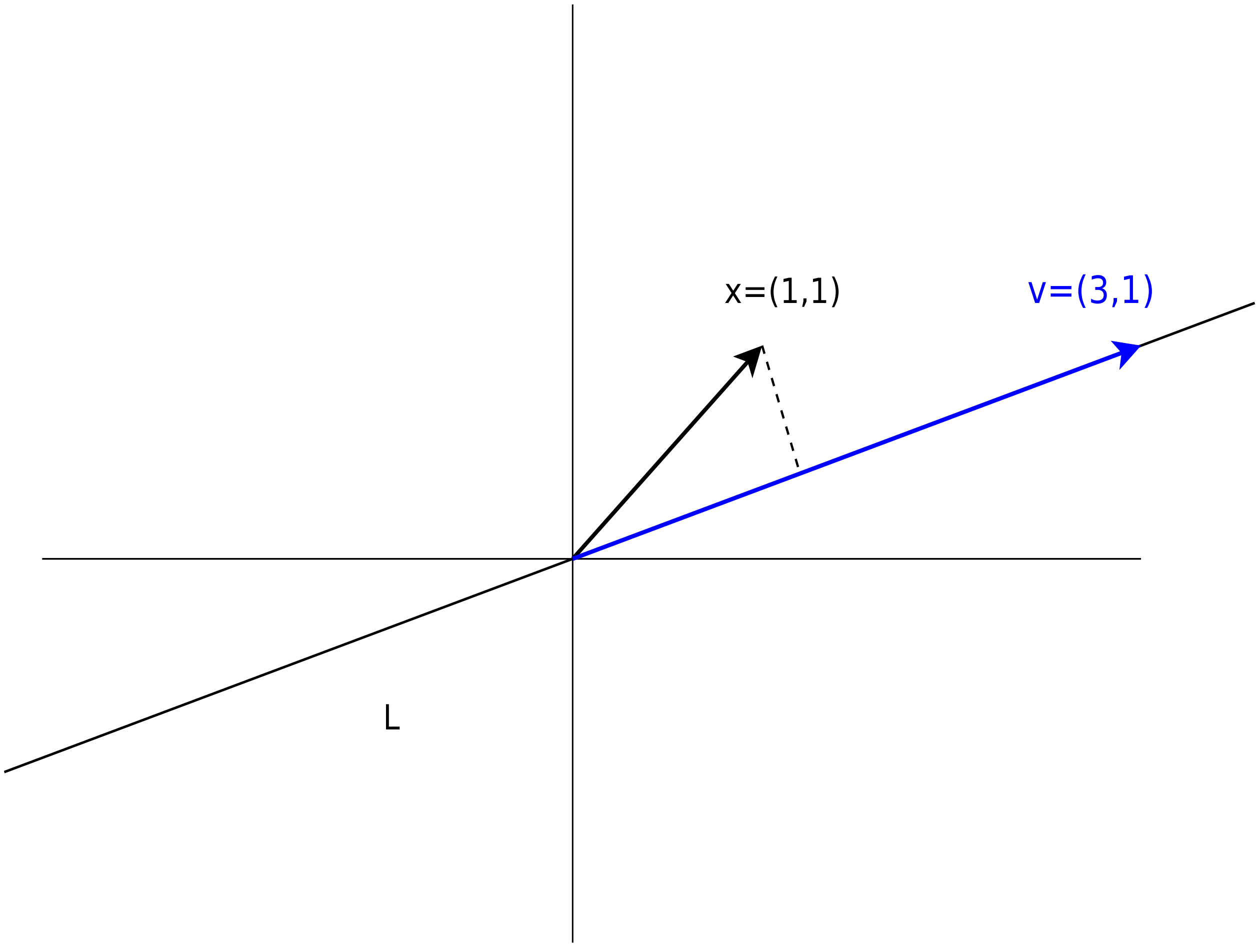
$$\text{proj}_L(\vec{w}) = A\vec{w} = \begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{4}w_1 + \frac{\sqrt{3}}{4}w_2 \\ \frac{\sqrt{3}}{4}w_1 + \frac{1}{4}w_2 \end{bmatrix}$$

(Remember, $\text{proj}_L(\vec{w})$ is a *vector*)

Projections

Example: Find the projection of the vector $\vec{x} = (1, 1)$ onto the line L that is parallel to the vector $\vec{v} = (3, 1)$.



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The procedure for finding the projection of a vector $\vec{x}$ onto a line L is:

- Find a unit vector $\vec{u}$ parallel to the line
- Compute the projection as: $\text{proj}_L(\vec{x}) \; = \; (\vec{x} \cdot \vec{u})\vec{u}$

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To convert $\vec{v}$ to a unit vector, we multiply it by the reciprocal of its length (which is a scalar). The length of $\vec{v}$ is:

$$\|\vec{v}\| \;=\; \sqrt{\vec{v} \cdot \vec{v}} \;=\; \sqrt{1^2 + 3^2} \;=\; \sqrt{10}$$

$$\vec{u} \;=\; \frac{1}{\|\vec{v}\|}\vec{v} \;=\; \frac{1}{\sqrt{10}}\vec{v}$$

Projections

or

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

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$$\vec{u} = \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{bmatrix}$$

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$$\vec{u} = \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{bmatrix}$$

Now

$$\text{proj}_L(\vec{w}) = (\vec{x} \cdot \vec{u})\vec{u} = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{bmatrix}\right)\begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{bmatrix}$$

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Expanding the dot product gives

$$\text{proj}_L(\vec{w}) = (\vec{x} \cdot \vec{u})\vec{u} = \left(1 \cdot \frac{1}{\sqrt{10}} + 1 \cdot \frac{3}{\sqrt{10}}\right) \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{bmatrix}$$

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$$\text{proj}_L(\vec{w}) = (\vec{x} \cdot \vec{u})\vec{u} = \left(\frac{4}{\sqrt{10}}\right) \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{bmatrix}$$

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Finally,

$$\text{proj}_L(\vec{w}) = (\vec{x}\cdot\vec{u})\vec{u} = \begin{bmatrix}\frac{4}{10}\\ \frac{12}{10}\end{bmatrix} = \begin{bmatrix}0.4\\ 1.2\end{bmatrix} = \vec{x}^{\parallel}$$

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The projection is

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