Matrix Algebra

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Introduction

The term **matrix algebra** refers to the *algebraic* properties of matrices and vectors:

- how the addition operation is defined
- how multiplication by scalars is defined
- which matrices or vectors can be added
- how the product of a vector and a matrix is defined

Matrices and Vectors

So far the following definitions have been presented:

- a **matrix** is a rectangular array of numbers
- a **column vector** is a matrix with a single column
- a **row vector** is a matrix with one row

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Vectors (either row or column) can be thought of as *elements of* $\mathbb{R}^n$, the set of ordered n-tuples of real numbers.

Examples

According to our definition, the following are matrices:

$$\begin{bmatrix} 1 & -4 & 0 \\ 2 & 2 & 7 \\ 1 & 0 & -11 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 6 \\ 3 & 2 & -2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -3 \\ 2 & 3 \\ 1 & 0 \\ 0 & 3 \end{bmatrix} \quad [1]$$

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The following are not matrices:

$$\begin{bmatrix} & -4 & 0 \\ 2 & 2 & 7 \\ 1 & 0 & -11 \end{bmatrix} \quad \begin{bmatrix} 1 & & & \\ 3 & 2 & -2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & \text{Ted} \\ 2 & 3 \\ 1 & 0 \\ 0 & 3 \end{bmatrix} \quad [\$]$$

Examples

According to our definition, the following are vectors (a vector is also a matrix):

$$\begin{bmatrix} 1 & -4 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \quad [1]$$

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The following are neither vectors nor matrices:

$$\begin{bmatrix} \& & -4 & 0 \end{bmatrix} \quad \begin{bmatrix} red & blue & magenta & cyan \end{bmatrix} \quad \begin{bmatrix} 1 \\ - \\ - \\ 0 \end{bmatrix}$$

Examples

If the a_{ij} symbols represent numbers, the following entity is a matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The a_{ij} are referred to as *entries*

Sums of Matrices

Addition of two matrices with the same numbers of rows and columns is defined entry by entry:

$$A + B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & c_{13} \\ b_{21} & b_{22} & c_{23} \\ b_{31} & b_{32} & c_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \\ a_{31} + b_{31} & a_{32} + b_{32} & a_{33} + b_{33} \end{bmatrix}$$

The matrix sum has the same number of rows and columns as each of the matrix addends.

Scalar Multiplication

Multiplication of a matrix by a scalar is also defined entry by entry:

$$kA = k \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} k \cdot a_{11} & k \cdot a_{12} & k \cdot a_{13} \\ k \cdot a_{21} & k \cdot a_{22} & k \cdot a_{23} \\ k \cdot a_{31} & k \cdot a_{32} & k \cdot a_{33} \end{bmatrix}$$

The resulting matrix has the same number of rows and columns as the original matrix.

Linear Combinations

A vector $\vec{b} \in \mathbb{R}^n$ is said to be a **linear combination** of the vectors

$$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \ \in \mathbb{R}^n$$

if we can find k scalars $a_1, a_2, \ldots, a_k$ such that

$$\vec{b} \ = \ a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k$$

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A row or column vector with n components is considered to be equivalent to an *ordered n-tuple* $(x_1, x_2, \ldots, x_n)$ of real numbers. The set of all *ordered n-tuples* of real numbers is denoted by $\mathbb{R}^n$.

Any element of $\mathbb{R}^n$ can be thought of as either a row vector or a column vector witn n components.

Matrix Columns as Vectors

It is very often useful to think of the columns of a matrix as column vectors:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$$

where

$$\vec{v}_1 = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} a_{21} \\ a_{22} \\ a_{23} \end{bmatrix} \quad \text{and} \quad \vec{v}_3 = \begin{bmatrix} a_{31} \\ a_{32} \\ a_{33} \end{bmatrix}$$

Matrix Rows as Vectors

Likewise is very often useful to think of the rows of a matrix as row vectors:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \end{bmatrix}$$

where

$$\vec{v}_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$\vec{v}_3 = \begin{bmatrix} a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The Product of a Matrix and a Vector

If A is an $n \times m$ matrix and a column vector $\vec{x}$ has m components, we may define the **product** $A\vec{x}$ as

$$A\vec{x} = [\vec{v}_1 \; \vec{v}_2 \cdots \vec{v}_m] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m$$

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The product

$$A\vec{x}$$

is a vector in $\mathbb{R}^n$ because it is a linear combination of the vectors comprising the columns of A, which each have n components.

The Product of a Matrix and a Vector

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On the right side of the above equation, the $x_i\vec{v}_i$ entries represent the *scalar* product of x_i, the i^{th} element of $\vec{x}$, with the column vector $\vec{v}_i$.

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The result of these m scalar multiplications is a set of m vectors $\vec{v_i} \in \mathbb{R}^n$, $i = 1, 2, \ldots, m$.

The Product of a Matrix and a Vector

$$A\vec{x} = [\vec{v}_1 \ \vec{v}_2 \cdots \vec{v}_m] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m$$

On the right side of the above equation, the $x_i\vec{v}_i$ entries represent the *scalar* product of x_i, the i^{th} element of $\vec{x}$, with the column vector $\vec{v}_i$.

The result of these m scalar multiplications is a set of m vectors $\vec{v_i} \in \mathbb{R}^n$, $i = 1, 2, \ldots, m$.

The final result $A\vec{x}$ is the sum of these m vectors in $\mathbb{R}^n$, and is an element of $\mathbb{R}^n$.

The Product of a Matrix and a Vector

Alternatively, we can view the $n \times m$ matrix A as n row vectors, each with m components.

In this case the product $A\vec{x}$ becomes

$$A\vec{x} \ = \begin{bmatrix} \vec{w}_1 \\ \vec{w}_2 \\ \vdots \\ \vec{w}_n \end{bmatrix} \vec{x} \ = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vec{w}_2 \cdot \vec{x} \\ \vdots \\ \vec{w}_n \cdot \vec{x} \end{bmatrix}$$

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The product

$$A\vec{x}$$

is a again vector in $\mathbb{R}^n$.

The Product of a Matrix and a Vector

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This time the $\vec{w}_i \cdot \vec{x}_i$ entries on right side of the above equation are *dot products* of two vectors, $\vec{w}_i$ and $\vec{x}$, which each have m components. There are n such products, one for each row of A.

The Product of a Matrix and a Vector

The **algebraic** rules that the product $A\vec{x}$ obeys are as follows:

Suppose A is an arbitrary $n \times m$ matrix, $\vec{x}$ and $\vec{y}$ are arbitrary vectors in $\mathbb{R}^m$, and k is an arbitrary scalar.

The following two equations hold:

$$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$$

$$A(k\vec{x}) = k\,(A\vec{x})$$

Matrix Form of a Linear System

We can now write any system of linear equations in matrix notation.

Suppose

$$\left[A \mid \vec{b}\right]$$

is the augmented matrix of a system of linear equations.

In matrix form, the system of linear equations is

$$A\vec{x} = \vec{b}$$

Matrix Form of a Linear System

Example: Suppose the matrix A and vector $\vec{b}$ given by

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 4 & 2 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

correspond to the augmented matrix $\left[A \mid \vec{b} \right]$ of a system of linear equations.

Matrix Form of a Linear System

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 4 & 2 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

Let $\vec{x}$ be the column vector

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

where x_1, x_2, and x_3 represent the variables in the linear system.

Matrix Form of a Linear System

The matrix form of the linear system is:

$$A\vec{x} = \vec{b}$$

which in this example translates to

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

Matrix Form of a Linear System

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

Expanding the left hand side, $A\vec{x}$, using the rules we defined for this type of product gives:

$$\begin{bmatrix} 1x_1 & + & 0x_2 & + & 3x_3 \\ 2x_1 & + & 4x_2 & + & 2x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$