Linear Transformations

Gene Quinn

Functions

First a review of the concept of a function is in order.

A formal definition of a function usually involves three entities:

- A set X called the *domain*
- A set Y which is sometimes called the *codomain*
- A **rule of assignment** that assigns a *unique* element of the codomain Y to every element of the domain X.

Functions

The bulk of one's early experience is usually with functions whose domain X and codomain Y are the real numbers, and the rule of assignment is a formula, say

$$y = x^2 + 2$$

Functions

In the study of linear algebra, we will be dealing with functions whose domain and codomain are sets of *vectors*.

We identify a vector $\vec{x}$ having n components with an *ordered n-tuple* of scalars (real numbers, in this course):

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

corresponds to the ordered n-tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

Linear Transformations

We will consider the class of functions T for which:

- the domain is the set of m-tuples of real numbers $\mathbb{R}^m$
- the codomain is the set of n-tuples of real numbers, $\mathbb{R}^n$

In mathematics, this situation is denoted by:

$$T : \mathbb{R}^m \rightarrow \mathbb{R}^n$$

Linear Transformations

Definition: A **linear transformation** is a function T that maps $\mathbb{R}^m$ into $\mathbb{R}^n$,

$$T : \mathbb{R}^m \rightarrow \mathbb{R}^n$$

in such a way that there exists an $n \times m$ matrix A with the property that

$$T(\vec{x}) = A\vec{x}$$

for every $\vec{x} \in \mathbb{R}^m$.

Linear Transformations

The following properties characterize linear transformations:

A transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is **linear** if and only if

$$\begin{aligned} T(\vec{u} + \vec{v}) \;&=\; T(\vec{u}) \;+\; T(\vec{v}) && \text{for all } \vec{u}, \vec{v} \in \mathbb{R}^m \\ T(k\vec{v}) \;&=\; kT(\vec{v}) && \text{for all } \vec{v} \in \mathbb{R}^m \text{ and all scalars } k \end{aligned}$$

The Matrix of a Linear Transformation

We have seen that every linear transformation $T : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is associated with an $n \times m$ matrix A with the property that

$$T(\vec{x}) \; = \; A\vec{x} \qquad \text{for every } \vec{x} \in \mathbb{R}^m$$

The Matrix of a Linear Transformation

We have seen that every linear transformation $T : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is associated with an $n \times m$ matrix A with the property that

$$T(\vec{x}) \;=\; A\vec{x} \qquad \text{for every } \vec{x} \in \mathbb{R}^m$$

As it turns out there is a relationship between the columns of the matrix A and the image under T of vectors of the form

$$\vec{e}_i \;=\; \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

The Matrix of a Linear Transformation

$$\vec{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

The vector $\vec{e}_i$ has every component zero except for the i^{th}, which is 1.

The Matrix of a Linear Transformation

The $n \times n$ matrix A associated with the linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is:

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_m) \end{bmatrix}$$

The Matrix of a Linear Transformation

The $n \times n$ matrix A associated with the linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is:

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_m) \end{bmatrix}$$

In other words,

- the first column of A is the vector in $\mathbb{R}^n$ that T maps $\vec{e}_1$ into
- the second column of A is the vector in $\mathbb{R}^n$ that T maps $\vec{e}_2$ into
- the third column of A is the vector in $\mathbb{R}^n$ that T maps $\vec{e}_3$ into
- the k^{th} column of A is the vector in $\mathbb{R}^n$ that T maps $\vec{e}_k$ into

The Matrix of a Linear Transformation

The vectors

$$\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_m \in \mathbb{R}^m$$

are sometimes called the **standard vectors** in $\mathbb{R}^m$.

The Matrix of a Linear Transformation

The vectors

$$\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_m \in \mathbb{R}^m$$

are sometimes called the **standard vectors** in $\mathbb{R}^m$.

In $\mathbb{R}^3$, the standard vectors $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ are usually denoted by $\vec{i}, \vec{j}, \vec{k}$.