Linear Independence

Gene Quinn

Linear Independence

The idea of *linear independence* plays a central role in linear algebra.

Linear Independence

The idea of *linear independence* plays a central role in linear algebra.

Linear independence is a property of a *collection* or *set* containing two or more vectors.

Linear Independence

The idea of *linear independence* plays a central role in linear algebra.

Linear independence is a property of a *collection* or *set* containing two or more vectors.

Our definition of linear independence requires that each of the vectors in the set have the same number of components.

Linear Independence

The idea of *linear independence* plays a central role in linear algebra.

Linear independence is a property of a *collection* or *set* containing two or more vectors.

Our definition of linear independence requires that each of the vectors in the set have the same number of components.

This is necessary because the definition involves addition of the vectors and addition is only defined for vectors with the same number of components.

Redundancy

The approach used in the text is to first introduce the idea of *redundancy*.

Redundancy

The approach used in the text is to first introduce the idea of *redundancy*.

If

$$V = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$$

is a set of vectors $\vec{v}_i \in \mathbb{R}^n$, we say that the vector $\vec{v}_j$ is **redundant** if $\vec{v}_j$ is a linear combination of the vectors with $\vec{v}_1, \ldots, \vec{v}_{j-1}$.

Redundancy

The approach used in the text is to first introduce the idea of *redundancy*.

If

$$V = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$$

is a set of vectors $\vec{v}_i \in \mathbb{R}^n$, we say that the vector $\vec{v}_j$ is **redundant** if $\vec{v}_j$ is a linear combination of the vectors with $\vec{v}_1, \ldots, \vec{v}_{j-1}$.

That is, if there exists scalars $c_1, \ldots, c_{j-1}$ with the property that

$$\vec{v}_j = c_1\vec{v}_1 + \cdots + c_{j-1}\vec{v}_{j-1}$$

Redundancy

Example: Consider the following set of vectors:

$$V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$$

Redundancy

Example: Consider the following set of vectors:

$$V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$$

$\vec{v}_3$ is redundant because it is a linear combination of $\vec{v}_1$ and $\vec{v}_2$:

$$\vec{v}_1 + 2\vec{v}_2 = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \vec{v}_3$$

Redundancy

Example: It's not always easy to tell if vectors are redundant. Suppose

$$V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ -4 \end{bmatrix} \right\}$$

Redundancy

Example: It's not always easy to tell if vectors are redundant. Suppose

$$V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ -4 \end{bmatrix} \right\}$$

$\vec{v}_3$ is redundant because it is a linear combination of $\vec{v}_1$ and $\vec{v}_2$:

$$2\vec{v}_1 - 3\vec{v}_2 = 2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 4 \end{bmatrix} = \vec{v}_3$$

Redundancy

In general, how do we determine whether a set has any redundant vectors?

Here is a procedure that always works:

First, form a matrix A whose columns are the vectors in the set V:

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \end{bmatrix}$$

Redundancy

In general, how do we determine whether a set has any redundant vectors?

Here is a procedure that always works:

First, form a matrix A whose columns are the vectors in the set V:

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \end{bmatrix}$$

Next, compute $rref(A)$.

If every one of the m columns of $rref(A)$ contains a leading 1, there are no redundant vectors in V.

Redundancy

Example: Suppose as before

$$V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ -4 \end{bmatrix} \right\}$$

Redundancy

Example: Suppose as before

$$V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ -4 \end{bmatrix} \right\}$$

Form a matrix A whose columns are the elements of V:

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 10 \\ 1 & -1 & 5 \\ 1 & 2 & -4 \end{bmatrix}$$

Redundancy

Compute the reduced row-echelon form $rref(A)$:

$$rref(A) = rref \begin{bmatrix} 2 & -2 & 10 \\ 1 & -1 & 5 \\ 1 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Redundancy

Compute the reduced row-echelon form $rref(A)$:

$$rref(A) = rref \begin{bmatrix} 2 & -2 & 10 \\ 1 & -1 & 5 \\ 1 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

If we associate the first column of A and $rref(A)$ with $\vec{v}_1$, the second with $\vec{v}_2$, and so on, the vector corresponding to the first column without a leading 1 is redundant.

Redundancy

Compute the reduced row-echelon form $rref(A)$:

$$rref(A) = rref\begin{bmatrix} 2 & -2 & 10 \\ 1 & -1 & 5 \\ 1 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

If we associate the first column of A and $rref(A)$ with $\vec{v}_1$, the second with $\vec{v}_2$, and so on, the vector corresponding to the first column without a leading 1 is redundant.

In this case, $\vec{v}_3$ is redundant because the first column without a leading 1 is column 3, which corresponds to $\vec{v}_3$. We can also tell from $rref(A)$ that $2\vec{v}_1 - 3\vec{V}_2 = \vec{v}_3$.