
Inverse Example

Gene Quinn

Matrix Inversion Example

Example: Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

Matrix Inversion Example

Example: Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

First form the augmented matrix by appending I_3 to A on the right:

$$\left[A \mid I_3 \right] = \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array} \right]$$

Matrix Inversion Example

Now perform row reduction on the **entire** augmented matrix:

$$\left[A \mid I_3 \right] = \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array} \right]$$

Subtract twice the first row from the second.

Matrix Inversion Example

Now perform row reduction on the **entire** augmented matrix:

$$\left[A \mid I_3 \right] = \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array} \right]$$

Subtract twice the first row from the second.

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -7 & -1 & -2 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array} \right]$$

Matrix Inversion Example

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -7 & -1 & -2 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array} \right]$$

Subtract three times the first row from the third.

Matrix Inversion Example

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -7 & -1 & -2 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array} \right]$$

Subtract three times the first row from the third.

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -7 & -1 & -2 & 1 & 0 \\ 0 & -10 & -5 & -3 & 0 & 1 \end{array} \right]$$

Matrix Inversion Example

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -7 & -1 & -2 & 1 & 0 \\ 0 & -10 & -5 & -3 & 0 & 1 \end{array} \right]$$

Multiply the second row by $-1/7$:

Matrix Inversion Example

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -7 & -1 & -2 & 1 & 0 \\ 0 & -10 & -5 & -3 & 0 & 1 \end{array} \right]$$

Multiply the second row by $-1/7$:

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & -10 & -5 & -3 & 0 & 1 \end{array} \right]$$

Matrix Inversion Example

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & -10 & -5 & -3 & 0 & 1 \end{array} \right]$$

Add -4 times the second row to the first row:

Matrix Inversion Example

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & -10 & -5 & -3 & 0 & 1 \end{array} \right]$$

Add -4 times the second row to the first row:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{10}{7} & -\frac{1}{7} & \frac{4}{7} & 0 \\ 0 & 1 & \frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & -10 & -5 & -3 & 0 & 1 \end{array} \right]$$

Matrix Inversion Example

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{10}{7} & -\frac{1}{7} & \frac{4}{7} & 0 \\ 0 & 1 & \frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & -10 & -5 & -3 & 0 & 1 \end{array} \right]$$

Add 10 times the second row to the third row:

Matrix Inversion Example

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{10}{7} & -\frac{1}{7} & \frac{4}{7} & 0 \\ 0 & 1 & \frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & -10 & -5 & -3 & 0 & 1 \end{array} \right]$$

Add 10 times the second row to the third row:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{10}{7} & -\frac{1}{7} & \frac{4}{7} & 0 \\ 0 & 1 & \frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & 0 & -\frac{25}{7} & -\frac{1}{7} & -\frac{10}{7} & 1 \end{array} \right]$$

Matrix Inversion Example

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{10}{7} & -\frac{1}{7} & \frac{4}{7} & 0 \\ 0 & 1 & \frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & 0 & -\frac{25}{7} & -\frac{1}{7} & -\frac{10}{7} & 1 \end{array} \right]$$

Multiply the third row by $-\frac{7}{25}$:

Matrix Inversion Example

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{10}{7} & -\frac{1}{7} & \frac{4}{7} & 0 \\ 0 & 1 & \frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & 0 & -\frac{25}{7} & -\frac{1}{7} & -\frac{10}{7} & 1 \end{array} \right]$$

Multiply the third row by $-\frac{7}{25}$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{10}{7} & -\frac{1}{7} & \frac{4}{7} & 0 \\ 0 & 1 & \frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & 0 & 1 & \frac{1}{25} & \frac{2}{5} & -\frac{7}{25} \end{array} \right]$$

Matrix Inversion Example

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{10}{7} & -\frac{1}{7} & \frac{4}{7} & 0 \\ 0 & 1 & \frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & 0 & 1 & \frac{1}{25} & \frac{2}{5} & -\frac{7}{25} \end{array} \right]$$

Add $-1/7$ times the third row to the second row:

Matrix Inversion Example

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{10}{7} & -\frac{1}{7} & \frac{4}{7} & 0 \\ 0 & 1 & \frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & 0 & 1 & \frac{1}{25} & \frac{2}{5} & -\frac{7}{25} \end{array} \right]$$

Add $-1/7$ times the third row to the second row:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{10}{7} & -\frac{1}{7} & \frac{4}{7} & 0 \\ 0 & 1 & 0 & \frac{7}{25} & -\frac{1}{5} & \frac{1}{25} \\ 0 & 0 & 1 & \frac{1}{25} & \frac{2}{5} & -\frac{7}{25} \end{array} \right]$$

Matrix Inversion Example

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{10}{7} & -\frac{1}{7} & \frac{4}{7} & 0 \\ 0 & 1 & 0 & \frac{7}{25} & -\frac{1}{5} & \frac{1}{25} \\ 0 & 0 & 1 & \frac{1}{25} & \frac{2}{5} & -\frac{7}{25} \end{array} \right]$$

Add $-10/7$ times the third row to the first row:

Matrix Inversion Example

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{10}{7} & -\frac{1}{7} & \frac{4}{7} & 0 \\ 0 & 1 & 0 & \frac{7}{25} & -\frac{1}{5} & \frac{1}{25} \\ 0 & 0 & 1 & \frac{1}{25} & \frac{2}{5} & -\frac{7}{25} \end{array} \right]$$

Add $-10/7$ times the third row to the first row:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{5} & 0 & \frac{2}{5} \\ 0 & 1 & 0 & \frac{7}{25} & -\frac{1}{5} & \frac{1}{25} \\ 0 & 0 & 1 & \frac{1}{25} & \frac{2}{5} & -\frac{7}{25} \end{array} \right]$$

Matrix Inversion Example

The augmented matrix now has the form:

$$\left[I_3 \mid B \right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{5} & 0 & \frac{2}{5} \\ 0 & 1 & 0 & \frac{7}{25} & -\frac{1}{5} & \frac{1}{25} \\ 0 & 0 & 1 & \frac{1}{25} & \frac{2}{5} & -\frac{7}{25} \end{array} \right]$$

Matrix Inversion Example

The augmented matrix now has the form:

$$\left[I_3 \mid B \right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{5} & 0 & \frac{2}{5} \\ 0 & 1 & 0 & \frac{7}{25} & -\frac{1}{5} & \frac{1}{25} \\ 0 & 0 & 1 & \frac{1}{25} & \frac{2}{5} & -\frac{7}{25} \end{array} \right]$$

Since the left half of the augmented matrix has become an identity matrix, we conclude that the right half is:

$$A^{-1} = B = \begin{bmatrix} -\frac{1}{5} & 0 & \frac{2}{5} \\ \frac{7}{25} & -\frac{1}{5} & \frac{1}{25} \\ \frac{1}{25} & \frac{2}{5} & -\frac{7}{25} \end{bmatrix}$$

Matrix Inversion Example

Let's check whether $B = A^{-1}$. Pick an arbitrary $\vec{v} \in \mathbb{R}^3$, say $(1, 1, 1)$, and form $T(\vec{v}) = A\vec{v}$:

$$A\vec{v} = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Matrix Inversion Example

Let's check whether $B = A^{-1}$. Pick an arbitrary $\vec{v} \in \mathbb{R}^3$, say $(1, 1, 1)$, and form $T(\vec{v}) = A\vec{v}$:

$$A\vec{v} = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The result is:

$$A\vec{v} = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 6 \end{bmatrix}$$

Matrix Inversion Example

Now multiply B on the right by $A\vec{v} = (7, 6, 6)$:

$$B(A\vec{v}) = \begin{bmatrix} -\frac{1}{5} & 0 & \frac{2}{5} \\ \frac{7}{25} & -\frac{1}{5} & \frac{1}{25} \\ \frac{1}{25} & \frac{2}{5} & -\frac{7}{25} \end{bmatrix} \begin{bmatrix} 7 \\ 6 \\ 6 \end{bmatrix}$$

Matrix Inversion Example

Now multiply B on the right by $A\vec{v} = (7, 6, 6)$:

$$B(A\vec{v}) = \begin{bmatrix} -\frac{1}{5} & 0 & \frac{2}{5} \\ \frac{7}{25} & -\frac{1}{5} & \frac{1}{25} \\ \frac{1}{25} & \frac{2}{5} & -\frac{7}{25} \end{bmatrix} \begin{bmatrix} 7 \\ 6 \\ 6 \end{bmatrix}$$

The result is:

$$B(A\vec{v}) = \begin{bmatrix} -\frac{1}{5} & 0 & \frac{2}{5} \\ \frac{7}{25} & -\frac{1}{5} & \frac{1}{25} \\ \frac{1}{25} & \frac{2}{5} & -\frac{7}{25} \end{bmatrix} \begin{bmatrix} 7 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

So apparently $B = A^{-1}$.

Matrix Inversion Example

We have shown that the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ associated with the matrix

$$T(\vec{v}) = A\vec{v} = \vec{u} \quad \text{where} \quad A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

is invertible and its inverse $T^{-1} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is associated with the matrix

$$T^{-1}(\vec{u}) = B\vec{u} = \vec{v} \quad \text{where} \quad B = \begin{bmatrix} -\frac{1}{5} & 0 & \frac{2}{5} \\ \frac{7}{25} & -\frac{1}{5} & \frac{1}{25} \\ \frac{1}{25} & \frac{2}{5} & -\frac{7}{25} \end{bmatrix}$$