Inverse of a Linear Transformation

Gene Quinn

Invertible Functions

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T and T^{-1} have the following properties:

$$T^{-1}(T(x)) = x \quad \forall x \in X \quad \text{and} \quad T(T^{-1}(y)) = y \quad \forall y \in Y$$

Invertible Linear Transformations

A linear transformation

$$T : \mathbb{R}^m \to \mathbb{R}^n \quad \text{defined by} \quad T(\vec{x}) = A\vec{x} = \vec{y}$$

for some $n \times m$ matrix A is invertible if and only if the following two conditions are satisfied:

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1) $n = m$, that is, $T : \mathbb{R}^n \to \mathbb{R}^n$.

2) The reduced row-echelon form of A is an identity matrix:

$$\text{rref}(A); = \ I_n \ = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Invertible Matrices

When a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible, the matrix A associated with T

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is said to be **invertible**.

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The matrix associated with the *inverse* of T,

$$T^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

is denoted by A^{-1}:

$$T^{-1}(\vec{y}) \;=\; A^{-1}\vec{y} = \vec{x}$$

Invertible Matrices

For an invertible linear transformation T, the relationships between T, T^{-1}, A, A^{-1}, $\vec{x}$, and $\vec{y}$ are as follows:

$$T(\vec{x}) = A\vec{x} = \vec{y}$$

$$T^{-1}(\vec{y}) = A^{-1}\vec{y} = \vec{x}$$

Invertible Matrices

All invertible matrices are square

However, not all square matrices are invertible.

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If A is not invertible, the system

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has either no solution or an infinite number of solutions.

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If A is invertible, $\vec{x} = \vec{0}$ is the *only* solution to this system.

If A is not invertible, the system has infinitely many solutions.

Inverse of a 2×2 Matrix

Suppose A is a 2×2 matrix.

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When $ad - bc \neq 0$,

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Finding the Inverse of a Matrix

To find the inverse of a matrix A, form the $n \times 2n$ augmented matrix

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If the left half of rref $\left[\begin{array}{c:c} A & I_n \end{array}\right]$ is not I_n, then A is not invertible.