Introduction and Terminology

Gene Quinn

Scalars and Vectors

We will begin by defining two entities, **scalars** and **vectors**

Scalars and Vectors

We will begin by defining two entities, **scalars** and **vectors**

We will define a **scalar** k to be simply a real number,

$$k \in \mathbb{R}$$

Scalars and Vectors

We will begin by defining two entities, **scalars** and **vectors**

We will define a **scalar** k to be simply a real number,

$$k \in \mathbb{R}$$

The elements of the set of scalars, being real numbers, inherit all of the properties of real numbers with respect to addition, multiplication, ordering, etc.

Vectors

The text defines a vector as a rectangular array of real numbers with a single column.

Vectors

The text defines a vector as a rectangular array of real numbers with a single column.

While it works and doesn't require any additional terminology, as soon as possible you should abandon this definition in favor of a definition based on ordered n-tuples of real numbers:

The notation

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$$

represents the n-fold Cartesian product of the set of real numbers $\mathbb{R}$ with itself.

Vectors

The *Cartesian product* of two sets A and B, denoted by $A \times B$, is defined to be the set of all **ordered pairs** (a, b) whose first element belongs to A and second element belongs to B.

Vectors

The *Cartesian product* of two sets A and B, denoted by $A \times B$, is defined to be the set of all **ordered pairs** (a, b) whose first element belongs to A and second element belongs to B.

In set-builder notation,

$$A \times B \;=\; \{(a, b) \;:\; a \in A,\, b \in B\}$$

Vectors

Example: If

$$A = \{1, 2\} \quad \text{and} \quad B = \{3, 4, 5\}$$

then

$$A \times B \; = \; \{(1,3),(1,4),(1,5),(2,3),(2,4),(2,5)\}$$

Vectors

Example: If

$$A = \{1, 2\} \quad \text{and} \quad B = \{3, 4, 5\}$$

then

$$A \times B \;=\; \{(1,3),(1,4),(1,5),(2,3),(2,4),(2,5)\}$$

If A and B are the same set, the shorthand notation

$$A \times A \;=\; A^2$$

is used. The main advantage of this notation is that it is convenient for constructs like

$$A \times A \times A \times A \times A \times A \times A \;=\; A^7$$

Vectors

In fact entities like

$$A \times A \times A \times A \times A \times A \times A \;=\; A^7$$

occur frequently, usually in the form

$$\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \;=\; \mathbb{R}^7$$

Vectors

In fact entities like

$$A \times A \times A \times A \times A \times A \times A \; = \; A^7$$

occur frequently, usually in the form

$$\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \; = \; \mathbb{R}^7$$

$\mathbb{R}^7$ is just shorthand notation for the set of all ordered 7-tuples of real numbers. In set-builder notation, this is

$$\mathbb{R}^7 \; = \; \{(v_1, v_2, v_3, v_4, v_5, v_6, v_7) \; : \; v_i \in \mathbb{R}, \; i = 1, 2, \ldots, 7\}$$

Vectors

We will identify each ordered 7-tuple of real numbers with a **vector**:

$$\vec{v} = (v_1, v_2, v_3, v_4, v_5, v_6, v_7)$$

Vectors

We will identify each ordered 7-tuple of real numbers with a **vector**:

$$\vec{v} \;=\; (v_1, v_2, v_3, v_4, v_5, v_6, v_7)$$

The individual elements of the order 7-tuple are *scalars* (i.e., real numbers), and are called the **components** of the vector $\vec{v}$.

Vectors

More generally, we will identify each ordered n-tuple of real numbers with a **vector**:

$$\vec{v} = (v_1, v_2, \ldots, v_n)$$

Vectors

More generally, we will identify each ordered n-tuple of real numbers with a **vector**:

$$\vec{v} = (v_1, v_2, \ldots, v_n)$$

The set consisting of all vectors with n components will be identified with set of all ordered n-tuples of real numbers $\mathbb{R}^n$, and we will denote the fact that a vector $\vec{v}$ belongs to this set with the notation

$$\vec{v} \in \mathbb{R}^n$$

Vectors

As it suits our purpose, we will sometimes represent the vector

$$\vec{v} \in \mathbb{R}^n \; = \; (v_1, v_2, \ldots, v_n)$$

as a rectangular array with a single column (a *column vector*):

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Vectors

At other times, we will find it more convenient to represent the vector

$$\vec{v} \in \mathbb{R}^n \; = \; (v_1, v_2, \ldots, v_n)$$

as a rectangular array with a single row (a *row vector*):

$$\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

Vectors

At other times, we will find it more convenient to represent the vector

$$\vec{v} \in \mathbb{R}^n \;=\; (v_1, v_2, \ldots, v_n)$$

as a rectangular array with a single row (a *row vector*):

$$\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

It's probably best to think of the vector $\vec{v} \in \mathbb{R}^n$ itself as the ordered n-tuple of real numbers, with no particular arrangement in terms of rows and columns implied.

Vectors

Algebraic manipulations will often require that we represent the vector $\vec{v} \in \mathbb{R}^n$ in either column vector or row vector form.

Vectors

Algebraic manipulations will often require that we represent the vector $\vec{v} \in \mathbb{R}^n$ in either column vector or row vector form.

Following the text, we will adopt the convention that when a vector $\vec{v} \in \mathbb{R}^n$ appears in an algebraic expression, we will assume it is represented in column vector form unless otherwise stated.

Vectors

Although we have introduced them in a completely abstract setting, vectors are commonly used in the sciences and other applications to represent quantities that have both an associated *magnitude* and a *direction*.

Vectors

Although we have introduced them in a completely abstract setting, vectors are commonly used in the sciences and other applications to represent quantities that have both an associated *magnitude* and a *direction*.

Examples of such quantities from physics are velocity and force.

Vectors are a natural choice for representing these phenomena.

Vectors

We will define the **length** of a vector as follows:

If $\vec{v} \in \mathbb{R}^n$ is a vector, the **length** of $\vec{v}$, denoted by $\|\vec{v}\|$, is defined to be the *scalar* :

$$\|\vec{v}\| \; = \; \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Vectors

We will define the **length** of a vector as follows:

If $\vec{v} \in \mathbb{R}^n$ is a vector, the **length** of $\vec{v}$, denoted by $\|\vec{v}\|$, is defined to be the *scalar* :

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

If we think of a vector $\vec{v}$ as an entity having an associated magnitude and direction, the length would represent the magnitude.

Vectors

We will define the **length** of a vector as follows:

If $\vec{v} \in \mathbb{R}^n$ is a vector, the **length** of $\vec{v}$, denoted by $\|\vec{v}\|$, is defined to be the *scalar* :

$$\|\vec{v}\| \;=\; \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

If we think of a vector $\vec{v}$ as an entity having an associated magnitude and direction, the length would represent the magnitude.

A vector $\vec{u}$ having

$$\|\vec{u}\| = 1$$

is called a **unit vector**.

Vectors

The **standard representation** of a vector $\vec{v} \in \mathbb{R}^n$ is a directed line segment or *arrow* connecting the origin, $(0, 0, \ldots, 0)$ to the point with coordinates $(v_1, v_2, \ldots, v_n)$, usually with an arrowhead drawn at the end that is not the origin. Here are some vectors in $\mathbb{R}^2$: