Determinants of Elementary Matrices

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Elementary Matrices

Recall that an **elementary matrix** is any matrix that can be obtained from the identity matrix I_n by *exactly one* of the following operations:

- Add a multiple of one row to another row
- Multiply a row by some constant k
- Interchange two rows

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We will use the Weierstrass definition of the determinant to establish the determinant of each of the three types of elementary matrices.

The Weierstrass Definition

Recall that, for a square matrix A the determinant function $\det(A)$ has three properties:

- $\det(A)$ is linear in each row of A.
- Interchanging two rows changes the sign of $\det(A)$.
- The determinant of the identity matrix $\det(I)$ is 1.

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For each positive integer n, there is exactly one function

$$\det(A) : \mathbb{R}^{n \times n} \to \mathbb{R} \;=\; f(a_{11}, \ldots, a_{nn})$$

that has these three properties.

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We obtain this type of elementary matrix by interchanging two rows of an identity matrix.

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Multiplying an arbitrary matrix A on the left by a matrix of this type will interchange the same two rows of A.

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Taken together, the two properties imply that the determinant of an elementary matrix of this type is -1.

Multiply a Row by a Constant

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When an arbitrary matrix A is multiplied on the left by this elementary matrix, the corresponding row of A is multiplied by k.

Multiply a Row by a Constant

The third Weierstrass property states that $\det(I) = 1$.

We may write I in terms of unit vectors e_i,

$$\det(I) = \det \begin{bmatrix} \vec{e}_1 \\ \vdots \\ \vec{e}_j \\ \vdots \\ \vec{e}_n \end{bmatrix} = 1$$

Multiply a Row by a Constant

The elementary matrix E is obtained by multiplying one row of the identity matrix I by a constant k:

$$E = \begin{bmatrix} \vec{e}_1 \\ \vdots \\ k \cdot \vec{e}_j \\ \vdots \\ \vec{e}_n \end{bmatrix}$$

Multiply a Row by a Constant

By the first Weierstrass property, linearity in the rows,

$$\det(E) = \det \begin{bmatrix} \vec{e}_1 \\ \vdots \\ k \cdot \vec{e}_j \\ \vdots \\ \vec{e}_n \end{bmatrix} = k \cdot \det \begin{bmatrix} \vec{e}_1 \\ \vdots \\ \vec{e}_j \\ \vdots \\ \vec{e}_n \end{bmatrix} = k \det(I) = k \cdot 1 = k$$

Add a Multiple of One Row to Another

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When an arbitrary matrix A is multiplied on the left by this elementary matrix, the corresponding row of A is multiplied by k and added to the corresponding other row of A.

Add a Multiple of One Row to Another

This elementary matrix E is obtained by multiplying one row of the identity matrix I by a constant k and adding it to another:

$$E = \begin{bmatrix} \vec{e}_1 \\ \vdots \\ \vec{e}_i \\ \vdots \\ \vec{e}_j + k\vec{e}_i \\ \vdots \\ \vec{e}_n \end{bmatrix}$$

Add a Multiple of One Row to Another

By the first Weierstrass property (linearity in the rows),

$$\det E = \det \begin{bmatrix} \vec{e}_1 \\ \vdots \\ \vec{e}_i \\ \vdots \\ \vec{e}_j + k\vec{e}_i \\ \vdots \\ \vec{e}_n \end{bmatrix} = \det \begin{bmatrix} \vec{e}_1 \\ \vdots \\ \vec{e}_i \\ \vdots \\ \vec{e}_j \\ \vdots \\ \vec{e}_n \end{bmatrix} + \det \begin{bmatrix} \vec{e}_1 \\ \vdots \\ \vec{e}_i \\ \vdots \\ k\vec{e}_i \\ \vdots \\ \vec{e}_n \end{bmatrix}$$

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The first matrix on the left is I, so its determinant is 1.

Add a Multiple of One Row to Another

$$\det \begin{bmatrix} \vec{e}_1 \\ \vdots \\ \vec{e}_i \\ \vdots \\ \vec{e}_j + k\vec{e}_i \\ \vdots \\ \vec{e}_n \end{bmatrix} = 1 + k \det \begin{bmatrix} \vec{e}_1 \\ \vdots \\ \vec{e}_i \\ \vdots \\ \vec{e}_i \\ \vdots \\ \vec{e}_n \end{bmatrix}$$

Add a Multiple of One Row to Another

$$\det \begin{bmatrix} \vec{e}_1 \\ \vdots \\ \vec{e}_i \\ \vdots \\ \vec{e}_j + k\vec{e}_i \\ \vdots \\ \vec{e}_n \end{bmatrix} = 1 + k \det \begin{bmatrix} \vec{e}_1 \\ \vdots \\ \vec{e}_i \\ \vdots \\ \vec{e}_i \\ \vdots \\ \vec{e}_n \end{bmatrix}$$

Since the matrix on the right has two identical rows, its determinant is zero.

Add a Multiple of One Row to Another

The final result is

$$\det \begin{bmatrix} \vec{e}_1 \\ \vdots \\ \vec{e}_i \\ \vdots \\ \vec{e}_j + k\vec{e}_i \\ \vdots \\ \vec{e}_n \end{bmatrix} = 1 + k \cdot 0 = 1$$

Summary

The three kinds of elementary matrices and their determinants are:

- Add a multiple of one row to another: $\det(E_1) = 1$
- Multiply a row by a constant k: $\det(E_2) = k$
- Exchange two rows: $\det(E_3) = -1$