Determinants of Elementary Matrices

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Elementary Matrices

Recall that an **elementary matrix** is any matrix that can be obtained from the identity matrix I_n by *exactly one* of the following operations:

- Add a multiple of one row to another row
- Multiply a row by some constant k
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We will use the Weierstrass definition of the determinant to establish the determinant of each of the three types of elementary matrices.

The Weierstrass Definition

Recall that, for a square matrix A the determinant function det(A) has three properties:

- det(A) is linear in each row of A.
- Interchanging two rows changes the sign of det(A).
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For each positive integer n, there is exactly one function

$$\det(A): \mathbb{R}^{n \times n} \to \mathbb{R} = f(a_{11}, \dots, a_{nn})$$

that has these three properties.

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We obtain this type of elementary matrix by interchanging two rows of an identity matrix.

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Multiplying an arbitrary matrix A on the left by a matrix of this type will interchange the same two rows of A.

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Taken together, the two properties imply that the determinant of an elementary matrix of this type is -1.

The second type of elementary matrix consists of an identity matrix with a single entry multiplied by a constant k.

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When an arbitrary matrix A is multiplied on the left by this elementary matrix, the corresponding row of A is multiplied by k.

The third Weierstrass property states that det(I) = 1.

We may write I in terms of unit vectors e_i ,

$$\det(I) = \det \begin{bmatrix} \vec{e_1} \\ \vdots \\ \vec{e_j} \\ \vdots \\ \vec{e_n} \end{bmatrix} = 1$$

The elementary matrix E is obtained by multiplying one row of the identity matrix I by a constant k:

$$E = \begin{bmatrix} \vec{e_1} \\ \vdots \\ k \cdot \vec{e_j} \\ \vdots \\ \vec{e_n} \end{bmatrix}$$

By the first Weierstrass property, linearity in the rows,

$$\det(E) = \det \begin{bmatrix} \vec{e_1} \\ \vdots \\ k \cdot \vec{e_j} \\ \vdots \\ \vec{e_n} \end{bmatrix} = k \cdot \det \begin{bmatrix} \vec{e_1} \\ \vdots \\ \vec{e_j} \\ \vdots \\ \vec{e_n} \end{bmatrix} = k \det(I) = k \cdot 1 = k$$

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When an arbitrary matrix A is multiplied on the left by this elementary matrix, the corresponding row of A is multiplied by k and added to the corresponding other row of A.

This elementary matrix E is obtained by multiplying one row of the identity matrix I by a constant k and adding it to another:

$$E = \begin{bmatrix} \vec{e_1} \\ \vdots \\ \vec{e_i} \\ \vec{e_j} \\ \vec{e_j} + k\vec{e_i} \\ \vdots \\ \vec{e_n} \end{bmatrix}$$

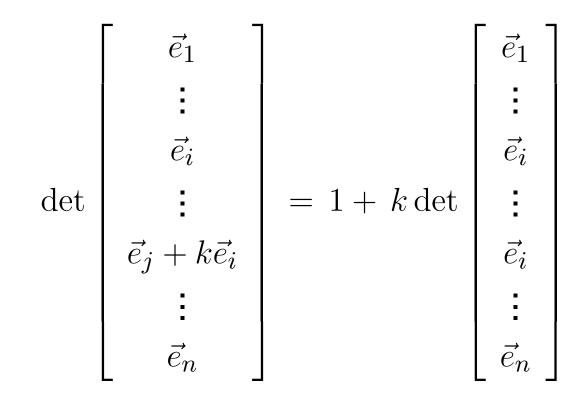
By the first Weierstrass property (linearity in the rows),

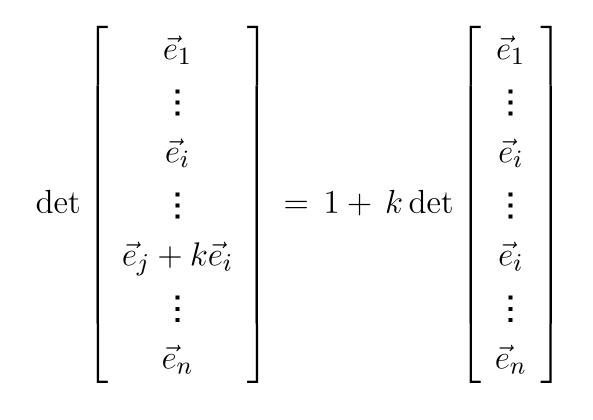
$$\det E = \det \begin{bmatrix} \vec{e}_{1} \\ \vdots \\ \vec{e}_{i} \\ \vdots \\ \vec{e}_{j} + k\vec{e}_{i} \\ \vdots \\ \vec{e}_{n} \end{bmatrix} = \det \begin{bmatrix} \vec{e}_{1} \\ \vdots \\ \vec{e}_{i} \\ \vdots \\ \vec{e}_{i} \end{bmatrix} + \det \begin{bmatrix} \vec{e}_{1} \\ \vdots \\ \vec{e}_{i} \\ \vdots \\ \vec{e}_{i} \\ \vdots \\ \vec{e}_{n} \end{bmatrix}$$

By the first Weierstrass property (linearity in the rows),

$$\det E = \det \begin{bmatrix} \vec{e_1} \\ \vdots \\ \vec{e_i} \\ \vec{e_i} \\ \vec{e_j} + k\vec{e_i} \\ \vdots \\ \vec{e_n} \end{bmatrix} = \det \begin{bmatrix} \vec{e_1} \\ \vdots \\ \vec{e_i} \\$$

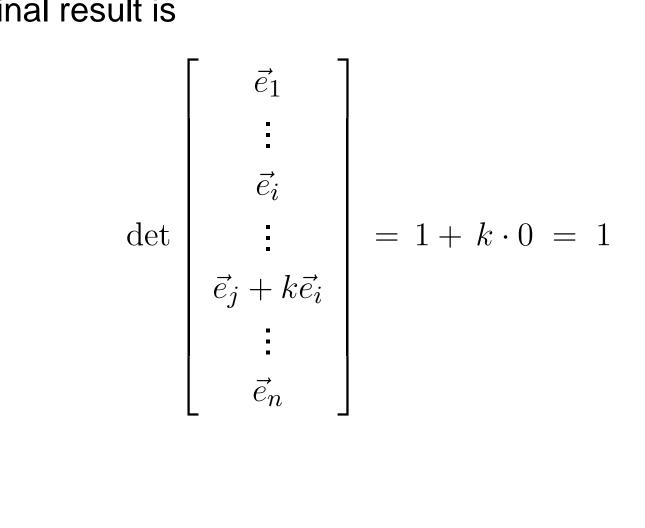
The first matrix on the left is *I*, so its determinant is 1.





Since the matrix on the right has two identical rows, its determinant is zero.

The final result is



Summary

The three kinds of elementary matrices and their determinants are:

- Add a multiple of one row to another: $det(E_1) = 1$
- Multiply a row by a constant k: $det(E_2) = k$
- Exchange two rows: $det(E_3) = -1$