Dot Products

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Definition

An important construct in linear algebra is the **dot product** of two vectors.

Suppose

$$\vec{v} \in \mathbb{R}^n = (v_1, v_2, \dots, v_n) \quad \text{and} \quad \vec{w} \in \mathbb{R}^n = (w_1, w_2, \dots w_n)$$

are vectors with n components each.

We define the **dot product** of $\vec{v}$ and $\vec{w}$, denoted by $\vec{v} \cdot \vec{w}$, as

$$\vec{v} \cdot \vec{w} \; = \; v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

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The dot product of two vectors $\vec{v} \cdot \vec{w}$ is a *scalar*

Algebraic Properties

The dot product has the following algebraic properties:

For arbitrary vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and arbitrary scalar $k \in \mathbb{R}$,

$$\begin{aligned}
\vec{v} \cdot \vec{w} &= \vec{w} \cdot \vec{v} \\
(\vec{u} + \vec{v}) \cdot \vec{w} &= \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w} \\
(k\vec{v}) \cdot \vec{w} &= k(\vec{v} \cdot \vec{w}) \\
\vec{v} \cdot \vec{v} &> 0 \quad \text{for all} \quad \vec{v} \neq \vec{0}
\end{aligned}$$

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It's not hard to verify these identities, and in fact doing so is a good exercise.

The Dot Product and Length

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Then

$$\vec{v} \cdot \vec{v} \;=\; v_1^2 + v_2^2 + \cdots + v_n^2 \;=$$

$$=\; \left(\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}\right)^2 \;=\; \|\vec{v}\|^2$$

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so

$$\|\vec{v}\| \; = \; \sqrt{\vec{v} \cdot \vec{v}}$$

The Dot Product and Length

The fact that

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$$

gives us the following technique for obtaining a unit vector $\vec{u}$ that points in the same direction as a given vector $\vec{v}$:

$$\vec{u} = \frac{\vec{v}}{\sqrt{\vec{v} \cdot \vec{v}}} = \frac{\vec{v}}{\|\vec{v}\|}$$

The Dot Product and Length

A well-known identity (which for some reason is not in the text) is the following: If θ is the angle between two vectors $\vec{v}$ and $\vec{w}$, then

$$\vec{v} \cdot \vec{w} \;=\; \|\vec{v}\|\|\vec{w}\| \cos\theta$$

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For example, if $\vec{v} \in \mathbb{R}^2 = (1, 0)$ and $\vec{w} \in \mathbb{R}^2 = (1, 1)$, then

$$\vec{v} \cdot \vec{w} \; = \; v_1 w_1 + v_2 w_2 = 1 \cdot 1 + 0 \cdot 1 \; = \; 1$$

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Note that $\|\vec{v}\| = 1$ and $\|\vec{w}\| = \sqrt{2}$.

The angle between $\vec{v}$ and $\vec{w}$ is $\pi/4$, so $\cos\theta = 1/\sqrt{2}$, and

$$\|\vec{v}\| \|\vec{w}\| \cos\theta = 1 \cdot \sqrt{2} \cdot \frac{1}{\sqrt{2}} = 1 = \vec{v} \cdot \vec{w}$$

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