THE DETERMINANT OF A DIAGONAL MATRIX

Definition:

An $n \times n$ matrix A with $(ij)^{th}$ entry a_{ij} is called:

upper triangular	if $a_{ij} = 0$	whenever	$i > j$
lower triangular	if $a_{ij} = 0$	whenever	$i < j$
diagonal	if $a_{ij} = 0$	whenever	$i \neq j$

Definition: (permutations and inversions definition of the determinant)

If A is an $n \times n$ matrix,

$$\det(A) = \sum_{all\ n-permutations} \pm a_{1j_1} \cdot a_{2j_2} \cdots a_{nj_n}$$

where

$$j_1,\, j_2,\, \ldots,\, j_n$$

is an n-permutation (i.e., a list of the first n positive integers written in any order) and the leading sign in each term is:

positive	if the number of inversions in the n-permutation $j_1, \ldots, j_n$ is even
negative	if the number of inversions in the n-permutation $j_1, \ldots, j_n$ is odd

and by an *inversion* we mean an ordered pair (j_m, j_k) consisting of two elements of the n-permutation

$$j_1,\, j_2,\, \ldots,\, j_n$$

having $m < k$ and $j_m > j_k$.

Theorem: The determinant of an $n \times n$ diagonal matrix A is the product of its diagonal entries:

$$\det(A) = \prod_{i=1}^{n} a_{ii}$$

Proof. By definition,

$$\det(A) \;=\; \sum_{all\ n-permutations} \pm a_{1j_1} \cdot a_{2j_2} \cdots a_{nj_n}$$

By hypothesis, A is diagonal, so

$$a_{ij} \;=\; 0 \quad \text{whenever} \quad i \neq j$$

Consequently, every nonzero term in the sum comprising $\det(A)$ must have:

$$j_1 = 1,\ j_2 = 2,\ j_3 = 3, \ldots, j_n = n$$

The only n-permutation that satisfies this condition is:

$$j_1, j_2, j_3, \ldots, j_n \;=\; 1, 2, 3, \ldots, n$$

which has zero inversions, so the expression for the determinant reduces to a single term:

$$\det(A) \;=\; a_{11} \cdot a_{22} \cdots a_{nn} \;=\; \prod_{i=1}^{n} a_{ii}$$

□