Defining the Determinant using Permutations

Gene Quinn

n-Permutations

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in any order.

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In general, the number of different n-permutations is

$$n! \ = \ n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$

n-Permutations

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Given the 4-permutation $3, 1, 2, 4$, we can construct

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4 \cdot 3}{2 \cdot 1} = 6$$

ordered pairs (preserving the order in the 4-permutation):

$$(3,1)\ (3,2)\ (3,4)\ (1,2)\ (1,4)\ (2,4);$$

n-Permutations

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We say that the 4-permutation $3, 1, 2, 4$ *has two inversions*.

n-Permutations

Among the 10 ordered pairs (a, b) constructed from the 5-permuation $3, 1, 5, 2, 4$,

$$(3,1)\ (3,5)\ (3,2)\ (3,4)\ (1,5)\ (1,2)\ (1,4)\ (5,2)\ (5,4)\ (2,4)$$

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We say that the 5-permutation $3, 1, 5, 2, 4$ *has four inversions*.

n-Permutations

For an $n \times n$ matrix A, consider the set of all possible products of the form

$$a_{1j_1} a_{2j_2} \cdots a_{nj_n}$$

where $j_1, j_2, \ldots, j_n$ is an n-permutation.

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Notice that this is the set of all products we can form by choosing n elements from A so that:

- exactly one element is chosen from each row
- exactly one element is chosen from each column

The Determinant and Permutations

Theorem: For any $n \times n$ matrix A,

$$\det(A) = \sum \pm a_{1j_1} a_{2j_2} \cdots a_{nj_n}$$

where:

- The sum is taken over all possible n-permutations $j_1, j_2, \ldots, j_n$
- The sign is taken to be $+$ if $j_1, j_2, \ldots, j_n$ has an even number of inversions
- The sign is taken to be $-$ if $j_1, j_2, \ldots, j_n$ has an odd number of inversions

The Determinant and Permutations

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The first has zero inversions, the second has one.

By the previous theorem, the determinant of a 2×2 matrix A is:

$$\det(A) \ = \ (+1)a_{11}a_{22} + (-1)a_{12}a_{21} \ = \ a_{11}a_{22} - a_{12}a_{21}$$

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This is equivalent to the formula $ad - bc$ we have been using.

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The inversions for the six 3-permutations are, respectively, 0,1,1,2,2, and 3, so the associated signs are $+, -, -, +, +, -$.

By the previous theorem, the determinant of a 3×3 matrix A is:

$$\begin{aligned} \det(A) \; &= \; (+1)a_{11}a_{22}a_{33} + (-1)a_{11}a_{23}a_{32} + (-1)a_{12}a_{21}a_{33} \\ &+ (+1)a_{12}a_{23}a_{31} + (+1)a_{13}a_{21}a_{32} + (-1)a_{13}a_{22}a_{31} \end{aligned}$$

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This formula is equivalent to Sarrus's rule (p. 248)