Computing Determinants with Gauss-Jordan Operations

Gene Quinn (from Carlos Curley's notes)

Computing Determinants

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$$\det(A) \;=\; \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} \;=\; ad - bc$$

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We now wish to extend this result to allow us to find determinants of larger (square) matrices.

Computing Determinants

We proved the following properties for determinants of 2×2 matrices:

- $\det(I_2) = 1$
- If A is upper triangular (or diagonal, or lower triangular), $\det(A)$ is the product of the diagonal elements of A.
- Adding a multiple of one row of A to another leaves $\det(A)$ unchanged
- Interchanging two rows changes the sign of $\det(A)$
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We also noted (without proof) that these properties hold for $n \times n$ matrices, $n = 2, 3, \ldots$.

Computing Determinants

Assuming the five properties from the previous slide hold for $n \times n$ matrices, we can develop a method of computing the determinant for any square matrix.

The method will be based on Gauss-Jordan or row operations.

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There are a number of (quite different) algorithms available for computing the determinant of an arbitrary square matrix A.

The method based on Gauss-Jordan operations is one of the most efficient in terms of the number of arithmetic operations required.

Computing Determinants

In outline form, the method is:

- Start with a square matrix A
- If necessary, perform row operations to transform it to an upper triangular matrix
- Keep track of each row operation and its effect on the determinant of the matrix
- When an upper triangular matrix is obtained, compute its determinant as the product of its diagonal entries
- Use the record of the changes to the determinant from the row operations to work backwards to the determinant of the original matrix A.

Elementary Matrices

Recall that the Gauss-Jordan reduction transforms a matrix A to $rref(A)$ by a series of operations on the rows of A.

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Each operation consists of one of the following kinds of row operations:

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- Multiply a row by some constant k
- Interchange two rows

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Recall that the Gauss-Jordan reduction transforms a matrix A to $rref(A)$ by a series of operations on the rows of A.

Each operation consists of one of the following kinds of row operations:

- Add a multiple of one row to another row
- Multiply a row by some constant k
- Interchange two rows

Although we considered them purely algebraic operations, in fact each of them is equivalent to multiplication of A on the left by a matrix.

Elementary Matrices

Definition: An **elementary matrix** is any matrix that can be obtained from the identity matrix I_n by *exactly one* of the following operations:

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We will examine this assertion in more detail.

Elementary Matrices

- Add a multiple of one row to another row

Start with an identity matrix

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

Elementary Matrices

Now for some $k \in \mathbb{R}$ add k times the second row to the first:

$$E_1 = \begin{bmatrix} 1 & k & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

(The result is an elementary matrix)

Elementary Matrices

Consider the effect of multiplying an arbitrary $n \times n$ matrix A (written as a column of row vectors) on the left by this matrix:

$$E_1 A = \begin{bmatrix} 1 & k & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 + k a_2 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

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We obtained the elementary matrix E_1 by adding k times the second row of I_n to the first row.

Multiplying A on the left by E_1 has the same effect on A.

Elementary Matrices

We now turn our attention to the determinant of the product $E_1 A$.

From the algebraic properties of determinants,

$$\det(E_1 A) \;=\; \det(E_1) \cdot \det(A)$$

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By a previous result (Property 2 of determinants), this means that $\det(E_1)$ is the product of the diagonal elements of E_1, which are all 1. Therefore,

$$\det(E_1) \;=\; 1 \quad \text{and} \quad \det(E_1 A) \;=\; \det(E_1) \cdot \det(A) \;=\; \det(A)$$

Elementary Matrices

- Multiply a row by some constant k

Once again, start with an identity matrix

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

Elementary Matrices

Now for some $k \in \mathbb{R}$ multiply the second row by k:

$$E_2 = \begin{bmatrix} 1 & & & \\ & k & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

(The result is another type of elementary matrix)

Elementary Matrices

Consider the effect of multiplying an arbitrary $n \times n$ matrix A (written as a column of row vectors) on the left by this matrix:

$$E_2A = \begin{bmatrix} 1 & & & \\ & k & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ ka_2 \\ \vdots \\ a_n \end{bmatrix}$$

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We obtained the elementary matrix E_2 by multiplying the second row of I_n by k.

Multiplying A on the left by E_2 has the same effect on A.

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Note that E_2 is diagonal.

By a previous result (Property 2 of determinants), this means that $\det(E_2)$ is the product of the diagonal elements of E_2, one of which is k and the rest are all 1. Therefore,

$\det(E_2) \;=\; k$ and $\det(E_2 A) \;=\; \det(E_2) \cdot \det(A) \;=\; k \cdot \det(A)$

Elementary Matrices

- Interchange two rows

Once again, start with an identity matrix

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

Elementary Matrices

Now interchange the first and second rows:

$$E_3 = \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

(The result is another type of elementary matrix)

Elementary Matrices

Consider the effect of multiplying an arbitrary $n \times n$ matrix A (written as a column of row vectors) on the left by this matrix:

$$E_3 A = \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_2 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

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We obtained the elementary matrix E_3 by interchanging the first two rows of I_n.

Multiplying A on the left by E_3 has the same effect on A.

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Now consider the determinant of the product $E_3 A$.

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By two previous results (Properties 1 and 4 of determinants), interchanging two rows reverses the sign of the determinant, and $\det(I_n) = 1$. This means that $\det(E_3) \;=\; -1$. Therefore,

$$\det(E_3) \;=\; -1 \quad \text{and} \quad \det(E_3A) \;=\; \det(E_3)\cdot\det(A) \;=\; -\det(A)$$

Computing the Determinant

Now we can write down our algorithm for using Gauss-Jordan reduction to compute a determinant.

Starting with an arbitrary $n \times n$ matrix A, we perform a series of row operations, each of which is equivalent to multiplication of A on the left by some elementary matrix E_i.

After the first row operation, we have a new matrix B_1, which is the product $E_1 A$:

$$E_1 A = B_1$$

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After the first row operation, we have a new matrix B_1, which is the product $E_1 A$:

$$E_1 A \;=\; B_1$$

After two row operations, we have the matrix B_2,

$$E_2 E_1 A \;=\; B_1$$

Computing the Determinant

Suppose after k row operations, the matrix B_k is upper triangular:

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Since the determinant of a matrix product is the product of the determinants,

$$\det(E_k) \cdots \det(E_2) \det(E_1) \det(A) \;=\; \det(B_k)$$

Computing the Determinant

At this point, since B_k is upper triangular,

- $\det(B_k)$ is the product of the diagonal elements of B_k.

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For each of the elementary matrices, we know the determinant as well:

- If E_i adds a multiple of one row to another, $\det(E_i) = 1$
- If E_i multiplies a row by a constant k, $\det(E_i) = k$
- If E_i interchanges two rows, $\det(E_i) = -1$

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So we can compute $\det(A)$ as

$$\det(A) = \frac{\det(B_k)}{\det(E_k) \cdots \det(E_2) \det(E_1)}$$

Computing the Determinant

Example: Compute the determinant of

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 1 & 3 & 1 \\ 3 & 3 & -11 \end{bmatrix}$$

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For the first row operation, exchange rows 1 and 2:

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_1 A = B_1 = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & -2 \\ 3 & 3 & -11 \end{bmatrix}$$

Note that $\det(E_1) = -1$

Computing the Determinant

For the second row operation, add -2 times the first row to the second:

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_2 E_1 A = B_2 = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -2 & -4 \\ 3 & 3 & -11 \end{bmatrix}$$

Note that $\det(E_2) = 1$

Computing the Determinant

For the third row operation, add -3 times the first row to the third:

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \quad E_3E_2E_1A = B_3 = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -2 & -4 \\ 0 & -6 & -14 \end{bmatrix}$$

Note that $\det(E_3) = 1$

Computing the Determinant

For the fourth row operation, multiply the second row by $-1/2$ to get a leading 1:

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_4 E_3 E_2 E_1 A = B_4 = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & -6 & -14 \end{bmatrix}$$

Note that $\det(E_4) = -1/2$

Computing the Determinant

For the fifth row operation, add 6 times the second row to the third row:

$$E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 1 \end{bmatrix}$$

$$E_5 E_4 E_3 E_2 E_1 A = B_5 = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

Note that $\det(E_5) = 1$

Computing the Determinant

Now since B_5 is upper triangular, no more row operations are required.

$$E_5E_4E_3E_2E_1A = B_5 = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

Computing the Determinant

Now since B_5 is upper triangular, no more row operations are required.

$$E_5 E_4 E_3 E_2 E_1 A \;=\; B_5 \;=\; \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

Note that $\det(B_5) = 1 \cdot 1 \cdot -2 \;=\; -2$

Now we can write

$$\det(E_5)\det(E_4)\det(E_3)\det(E_2)\det(E_1)\det(A) \;=\; \det(B_5)$$

Computing the Determinant

So

$$\det(A) = \frac{\det(B_5)}{\det(E_5)\det(E_4)\det(E_3)\det(E_2)\det(E_1)}$$

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Plugging in the values for the determinants,

$$\det(A) = \frac{-2}{(1)\cdot\left(\frac{-1}{2}\right)(1)\,(1)\,(-1)} = \frac{-2}{\frac{1}{2}} = -4$$

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For the first row operation, add -4 times the first row to the second:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_1 A = B_1 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 5 & 11 & 15 \end{bmatrix}$$

Note that $\det(E_1) = 1$

Computing the Determinant

For the second row operation, add -5 times the first row to the third:

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \quad E_2 E_1 A = B_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Note that $\det(E_2) = 1$

Computing the Determinant

For the third row operation, add -1 times the second row to the third:

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad E_3E_2E_1A = B_3 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that $\det(E_3) = 1$

Computing the Determinant

No more row operations are required because B_3 is upper triangular, and so the determinant is the product of the diagonal entries:

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Since

$$E_3 E_2 E_1 A \;=\; B_3$$

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Since

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$$\det(E_3)\det(E_2)\det(E_1)\det(A) \;=\; \det(B_3)$$

$$\det(A) \;=\; \frac{\det(B_3)}{\det(E_3)\det(E_2)\det(E_1)} \;=\; \frac{0}{1 \cdot 1 \cdot 1} \;=\; 0$$

Computing the Determinant

Note that

$$B_3 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so

$$rref(A) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I_3$$

which indicates that A does not have an inverse.

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This is consistent with the fact that $\det(A) = 0$.