Development of 2x2 Determinants

Gene Quinn (from Carlos Curley's notes)

Inverses

A real number $A \in \mathbb{R}$ has a *multiplicative* inverse if and only if:

$$a \neq 0$$

in which case $a^{-1} = 1/a$.

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We would like to generalize this concept to the set of square matrices:

We want to associate a single real number d with a square matrix in such a way that A has a *matrix* inverse if and only if:

$$d \neq 0$$

Determinants

More precisely, for an $n \times n$ square matrix A, we want to find a real-valued *function* of the n^2 real numbers that comprise A:

$$f \; : \; \mathbb{R}^{n \times n} \rightarrow \mathbb{R} \; = \; f(a_{11}, \ldots, a_{nn})$$

which we will call the **determinant** of A and denote by $\det(A)$, with the following property:

A has an inverse if and only if

$$\det(A) \; \neq \; 0$$

The 2×2 Case

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Suppose A is an *invertible* 2×2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The 2×2 Case

Now let's find A^{-1}.

As usual, we form an augmented matrix with A and I_2,

$$\left[\begin{array}{cc:cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

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As usual, we form an augmented matrix with A and I_2,

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

We now perform row operations to transform the left half of the augmented matrix to I_2, and the right half becomes A^{-1}.

The 2×2 Case

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

Assuming $a \neq 0$, we multiply the first row by $1/a$ to get a leading 1:

$$\left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right]$$

The 2×2 Case

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(If $a = 0$, we have to interchange rows and divide by c)

The 2×2 Case

Now add $-c$ times the first row to the second:

$$\left[\begin{array}{cc:cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right]$$

The 2×2 Case

Now add $-c$ times the first row to the second:

$$\left[\begin{array}{cc:cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right]$$

The result is:

$$\left[\begin{array}{cc:cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d - c\frac{b}{a} & -c\frac{1}{a} & 1 \end{array}\right]$$

The 2×2 Case

With a bit of simplification,

$$\left[\begin{array}{cc:cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d - c\frac{b}{a} & -c\frac{1}{a} & 1 \end{array}\right]$$

becomes

$$\left[\begin{array}{cc:cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{a} & \frac{-c}{a} & 1 \end{array}\right]$$

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Normally at this point we multiply the second row by the reciprocal of

$$\frac{ad - bc}{a}$$

to obtain a leading 1.

The 2×2 Case

However, note that if

$$\frac{ad - bc}{a} = 0 \quad \Rightarrow \quad ad - bc = 0$$

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it is impossible to do this.

In this case, we can never reduce the left half of the augmented matrix to I_2.

The 2×2 Case

Assuming $ad - bc \neq 0$, we multiply the second row by $a/(ad - bc)$ to get a leading 1:

$$\left[\begin{array}{cc:cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{a} & \frac{-c}{a} & 1 \end{array}\right]$$

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The result is

$$\left[\begin{array}{cc:cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

The 2×2 Case

Finally we complete the reduction by adding $-b/a$ times the second row to the first:

$$\left[\begin{array}{cc:cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

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The right half of the augmented matrix is now A^{-1}:

$$\left[\begin{array}{cc:cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

The 2×2 Case

Since we can only complete the reduction of a 2×2 matrix A when $ad - bc = 0$, we can say that:

A 2×2 *matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

has an inverse if and only if

$$ad - bc = 0$$

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In this case,

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The 2×2 Case

In light of the previous example for a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

we define the **determinant** of A, denoted by $\det(A)$, to be

$$\det(A) = ad - bc$$

Properties of the Determinant

Property 1:

$$\det(I_2) = \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

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$$\det(I_2) \;=\; \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \;=\; 1$$

Proof of Property 1: Directly from the definition of $\det(A)$, if

$$A \;=\; \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

then

$$\det(A) \;=\; ad - bc \;=\; 1 \cdot 1 - 0 \cdot 0 \;=\; 1$$

Properties of the Determinant

Property 2: If a 2×2 matrix A is upper triangular,

$$\det(A) = \det \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = ad$$

(The determinant is the product of the diagonal entries.)

Properties of the Determinant

Property 2: If a 2×2 matrix A is upper triangular,

$$\det(A) \;=\; \det \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \;=\; ad$$

(The determinant is the product of the diagonal entries.)
Proof of Property 2: Directly from the definition of $\det(A)$, if

$$A \;=\; \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

then

$$\det(A) \;=\; ad - bc \;=\; ad - b \cdot 0 \;=\; ad$$

Properties of the Determinant

Note that the determinant is also the product of the diagonal entries when A is lower triangular,

$$\det \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} = ad$$

or diagonal:

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The proof of these statements is similar to the proof for the upper triangular case.

Properties of the Determinant

Property 3: If a 2×2 matrix B is obtained from a 2×2 matrix A by adding a multiple of the first row to the second, then the determinants are the same:

$$\det(B) \ = \ \det(A)$$

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Proof of Property 3: Let

$$A \; = \; \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Properties of the Determinant

Form B by adding k times the first row of A to the second:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$B = \begin{bmatrix} a & b \\ c + ka & d + kb \end{bmatrix}$$

Properties of the Determinant

Form B by adding k times the first row of A to the second:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$B = \begin{bmatrix} a & b \\ c + ka & d + kb \end{bmatrix}$$

Directly from the definition,

$$\det(B) = \det \begin{bmatrix} a & b \\ c + ka & d + kb \end{bmatrix} = a(d + kb) - b(c + ka)$$

$$= ad + abk - bc - abk = ad - bc = \det(A)$$

Properties of the Determinant

A similar argument can be used to establish that adding a multiple of the second row to the first leaves the determinant unchanged.

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So, we may make the more general statement:

Property 3: If a 2×2 matrix B is obtained from a 2×2 matrix A by adding a multiple of one row to another, the determinant is unchanged; that is,

$$\det(B) \; = \; \det(A)$$

Properties of the Determinant

Property 4: If a 2×2 matrix B is obtained by interchanging the rows of a 2×2 matrix A,

$$\det(B) = -\det(A)$$

(Interchanging rows changes the sign of the determinant)

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(Interchanging rows changes the sign of the determinant)
Proof of Property 4: Directly from the definition, if

$$A \;=\; \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B \;=\; \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

then

$$\det(B) \;=\; cb - ad \;=\; -(ad - bc) \;=\; -\det(A)$$

Properties of the Determinant

Property 5: If a 2×2 matrix B is obtained by multiplying a row of a 2×2 matrix A by a constant k then,

$$\det(B) \ = \ k \cdot \det(A)$$

(Multiplying a row by k multiplies the determinant by k)

Properties of the Determinant

Property 5: If a 2×2 matrix B is obtained by multiplying a row of a 2×2 matrix A by a constant k then,

$$\det(B) \;=\; k \cdot \det(A)$$

(Multiplying a row by k multiplies the determinant by k)
Proof of Property 5: Directly from the definition, if

$$A \;=\; \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B \;=\; \begin{bmatrix} ka & kb \\ c & d \end{bmatrix}$$

then

$$\det(B) \;=\; adk - bck \;=\; k(ad - bc) \;=\; k \cdot \det(A)$$

Properties of the Determinant

A similar argument will show the same result if the second row is multiplied by k.

Determinants of Higher Order Matrice

The results we obtained for 2×2 matrices generalize to $n \times n$ matrices.

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More precisely,

- It is possible to define $\det(A)$ for a square matrix of any dimension n (The exact definition depends on n)
- Properties 1-5 of $\det(A)$ hold for $n = 2, 3, 4 \ldots$

Summary

For a square matrix A, the following statements are equivalent:

- $\det(A) \neq 0$
- A has an inverse
- The system $Ax = b$ has exactly one solution for any vector $b \in \mathbb{R}^n$ (including $\vec{0}$)
- $rref(A) = I$

Summary

For a square matrix A, the following statements are equivalent:

- $\det(A) = 0$
- A does not have an inverse
- The system $Ax = 0$ has many solutions

Summary

Determinants of $n \times n$ matrices obey the following algebraic rules:

- $\det(AB) = \det(A) \cdot \det(B)$
- $\det(A + B)$ may not equal $\det(A)$+$\det(B)$
- $\det(A^{-1}) \; = \; 1/\det(A)$
- $\det(cA) \; = \; c^n \det(A)$ for any constant c