The Weierstrass Definition

Gene Quinn

The Weierstrass Definition

We have seen that for a square matrix A, $\det(A)$ can be defined as either:

- The Laplace expansion down the first column
- The sum of all n-permutations of the elements with suitably defined signs

The Weierstrass Definition

We have seen that for a square matrix A, $\det(A)$ can be defined as either:

- The Laplace expansion down the first column
- The sum of all n-permutations of the elements with suitably defined signs

It is a fact that both definitions produce the same result.

Note that each of them defines $\det(A)$ by specifying how to compute it.

The Weierstrass Definition

There is another way to arrive at a definition of $\det(A)$ that does not involve a computational algorithm.

The Weierstrass Definition

There is another way to arrive at a definition of $\det(A)$ that does not involve a computational algorithm.

The **Weierstrass definition** starts with the assumption that, if the determinant associates a real number with every $n \times n$ matrix A, it can be thought of as a real-valued function of the n^2 elements of A:

$$\det(A) \,:\, \mathbb{R}^{n\times n} \rightarrow \mathbb{R} \,=\, f(a_{11}, \ldots, a_{nn})$$

The Weierstrass Definition

There is another way to arrive at a definition of $\det(A)$ that does not involve a computational algorithm.

The **Weierstrass definition** starts with the assumption that, if the determinant associates a real number with every $n \times n$ matrix A, it can be thought of as a real-valued function of the n^2 elements of A:

$$\det(A) \, : \, \mathbb{R}^{n \times n} \rightarrow \mathbb{R} \; = \; f(a_{11}, \ldots, a_{nn})$$

$\det(A)$ is defined by specifying the characteristics this function should have.

The Weierstrass Definition

As it turns out, only three characteristics are required:

For an $n \times n$ matrix A,

- $\det(A)$ is linear in the rows of A

The Weierstrass Definition

As it turns out, only three characteristics are required:

For an $n \times n$ matrix A,

- $\det(A)$ is linear in the rows of A
- Interchanging two rows changes the sign of $\det(A)$

The Weierstrass Definition

As it turns out, only three characteristics are required:

For an $n \times n$ matrix A,

- $\det(A)$ is linear in the rows of A
- Interchanging two rows changes the sign of $\det(A)$
- $\det(I_n) = 1$

The Weierstrass Definition

As it turns out, only three characteristics are required:

For an $n \times n$ matrix A,

- $\det(A)$ is linear in the rows of A
- Interchanging two rows changes the sign of $\det(A)$
- $\det(I_n) = 1$

For each positive integer n, there is exactly one function with these three properties.

The Weierstrass Definition

As it turns out, only three characteristics are required:

For an $n \times n$ matrix A,

- $\det(A)$ is linear in the rows of A
- Interchanging two rows changes the sign of $\det(A)$
- $\det(I_n) = 1$

For each positive integer n, there is exactly one function with these three properties.

As a result, these characteristics can be used to *define* $\det(A)$

The Weierstrass Definition

As it turns out, only three characteristics are required:

For an $n \times n$ matrix A,

- $\det(A)$ is linear in the rows of A
- Interchanging two rows changes the sign of $\det(A)$
- $\det(I_n) = 1$

For each positive integer n, there is exactly one function with these three properties.

As a result, these characteristics can be used to *define* $\det(A)$

We'll examine them in more detail.

The Weierstrass Definition

We can always write an $n \times n$ matrix A in terms of its rows,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix}$$

The Weierstrass Definition

We can always write an $n \times n$ matrix A in terms of its rows,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix}$$

Here $\vec{a}_i$ is a row vector representing the i^{th} row of A:

$$\vec{a}_i = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}$$

The Weierstrass Definition

We can always write an $n \times n$ matrix A in terms of its rows,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix}$$

Here $\vec{a}_i$ is a row vector representing the i^{th} row of A:

$$\vec{a}_i = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}$$

For example,

$$\vec{a}_1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$$

The Weierstrass Definition

Recall that the definitive characteristics of a linear transformation

$$T : \mathbb{R}^m \to \mathbb{R}^n$$

are that, for any $\vec{x}, \vec{y} \in \mathbb{R}^m$ and any $k \in \mathbb{R}$,

- $T(\vec{x} + \vec{y}) \;=\; T(\vec{x}) + T(\vec{y})$
- $T(k\vec{x}) \;=\; kT(\vec{x})$

The Weierstrass Definition

A real-valued *function* f

$$f : \mathbb{R}^n \rightarrow \mathbb{R}$$

is said to be linear if, for any $\vec{x}, \vec{y} \in \mathbb{R}^n$ and any $k \in \mathbb{R}$,

- $f(\vec{x} + \vec{y}) \;=\; f(\vec{x}) + f(\vec{y})$
- $f(k\vec{x}) \;=\; kf(\vec{x})$

The Weierstrass Definition

Again, considering A as a column of row vectors,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix}$$

we can think of $\det(A)$ as the *function*

$$\det(A) : \mathbb{R}^{n \times n} \to \mathbb{R} = f(\vec{a}_1, \ldots, \vec{a}_n)$$

The Weierstrass Definition

When we say that $\det(A)$ is *linear in the first row of* A we mean two things:

First, for any vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, if $\vec{a}_1 = \vec{u} + \vec{v}$,

$$\det \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix} = \det \begin{bmatrix} \vec{u} + \vec{v} \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix} = \det \begin{bmatrix} \vec{u} \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix} + \det \begin{bmatrix} \vec{v} \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix}$$

The Weierstrass Definition

When we say that $\det(A)$ is *linear in the first row of* A we mean two things:

First, for any vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, if $\vec{a}_1 \;=\; \vec{u} + \vec{v}$,

$$\det \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix} \;=\; \det \begin{bmatrix} \vec{u} + \vec{v} \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix} \;=\; \det \begin{bmatrix} \vec{u} \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix} \;+\; \det \begin{bmatrix} \vec{v} \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix}$$

In the equivalent function notation, this is:

$$f(\vec{u} + \vec{v}, \vec{a}_2, \ldots, \vec{a}_n) \;=\; f(\vec{u}, \vec{a}_2, \ldots, \vec{a}_n) \;+\; f(\vec{v}, \vec{a}_2, \ldots, \vec{a}_n)$$

The Weierstrass Definition

Second, the statement that $\det(A)$ is linear in the first row of A means that, for any vector $\vec{u} \in \mathbb{R}^n$ and any scalar $k \in \mathbb{R}$, if $\vec{a}_1 = k\vec{u}$,

$$\det \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix} = \det \begin{bmatrix} k\vec{u} \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix} = k \cdot \det \begin{bmatrix} \vec{u} \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix}$$

The Weierstrass Definition

Second, the statement that $\det(A)$ is linear in the first row of A means that, for any vector $\vec{u} \in \mathbb{R}^n$ and any scalar $k \in \mathbb{R}$, if $\vec{a}_1 = k\vec{u}$,

$$\det \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix} = \det \begin{bmatrix} k\vec{u} \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix} = k \cdot \det \begin{bmatrix} \vec{u} \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix}$$

In the equivalent function notation, this is:

$$f(k\vec{u}, \vec{a}_2, \ldots, \vec{a}_n) = k \cdot f(\vec{u}, \vec{a}_2, \ldots, \vec{a}_n)$$

The Weierstrass Definition

More generally, when we say that $\det(A)$ is *linear in the rows of* A we mean two things:

First, for any vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, and any positive integer $j, \quad 1 \leq j \leq n$, if $\vec{a}_j \; = \; \vec{u} + \vec{v}$,

$$\det \begin{bmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_j \\ \vdots \\ \vec{a}_n \end{bmatrix} = \det \begin{bmatrix} \vec{a}_1 \\ \vdots \\ \vec{u} + \vec{v} \\ \vdots \\ \vec{a}_n \end{bmatrix} = \det \begin{bmatrix} \vec{a}_1 \\ \vdots \\ \vec{u} \\ \vdots \\ \vec{a}_n \end{bmatrix} + \det \begin{bmatrix} \vec{a}_1 \\ \vdots \\ \vec{v} \\ \vdots \\ \vec{a}_n \end{bmatrix}$$

The Weierstrass Definition

In the equivalent function notation, this is:

$$f(\vec{a}_1, \ldots, \vec{a}_j, \ldots, \vec{a}_n) \;=\; f(\vec{a}_1, \ldots, \vec{u} + \vec{v}, \ldots, \vec{a}_n) \;=$$

$$f(\vec{a}_1, \ldots, \vec{u}, \ldots, \vec{a}_n) \;+\; f(\vec{a}_1, \ldots, \vec{v}, \ldots, \vec{a}_n)$$

The Weierstrass Definition

Second, for any vector $\vec{u} \in \mathbb{R}^n$ and any scalar $k \in \mathbb{R}$, and any positive integer $j, \quad 1 \le j \le n$, if $\vec{a}_j \ = \ k\vec{u}$,

$$\det \begin{bmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_j \\ \vdots \\ \vec{a}_n \end{bmatrix} \ = \ \det \begin{bmatrix} \vec{a}_1 \\ \vdots \\ k\vec{u} \\ \vdots \\ \vec{a}_n \end{bmatrix} \ = \ k \cdot \det \begin{bmatrix} \vec{a}_1 \\ \vdots \\ \vec{u} \\ \vdots \\ \vec{a}_n \end{bmatrix}$$

The Weierstrass Definition

In the equivalent function notation, this is:

$$f(\vec{a}_1, \ldots, \vec{a}_j, \ldots, \vec{a}_n) = f(\vec{a}_1, \ldots, k\vec{u}, \ldots, \vec{a}_n) =$$

$$k \cdot f(\vec{a}_1, \ldots, \vec{u}, \ldots, \vec{a}_n)$$

The Weierstrass Definition

To summarize, the first condition in the Weierstrass definition of the determinant is that the determinant function is linear in the rows of A.

The Weierstrass Definition

To summarize, the first condition in the Weierstrass definition of the determinant is that the determinant function is linear in the rows of A.

In function notation, if f is the determinant function,

$$f(\vec{a}_1, \ldots, \vec{u} + \vec{v}, \ldots, \vec{a}_n) \ =$$

$$f(\vec{a}_1, \ldots, \vec{u}, \ldots, \vec{a}_n) \ + \ f(\vec{a}_1, \ldots, \vec{v}, \ldots, \vec{a}_n)$$

and

$$f(\vec{a}_1, \ldots, k\vec{u}, \ldots, \vec{a}_n) \ = \ k \cdot f(\vec{a}_1, \ldots, \vec{u}, \ldots, \vec{a}_n)$$

The Weierstrass Definition

The second condition in the Weierstrass definition says that interchanging two rows changes the sign of $\det(A)$.

The Weierstrass Definition

The second condition in the Weierstrass definition says that interchanging two rows changes the sign of $\det(A)$.

Again writing A as a column of row vectors, this means that for positive integers i, j with $1 \leq i, j \leq n$ and $i \neq j$,

$$\det \begin{bmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_i \\ \vdots \\ \vec{a}_j \\ \vdots \\ \vec{a}_n \end{bmatrix} = -\det \begin{bmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_j \\ \vdots \\ \vec{a}_i \\ \vdots \\ \vec{a}_n \end{bmatrix}$$

The Weierstrass Definition

The third and final condition in the Weierstrass definition says that for any positive integer n, $\det(I_n) = 1$.

The Weierstrass Definition

The third and final condition in the Weierstrass definition says that for any positive integer n, $\det(I_n) = 1$.

Recall that in $\mathbb{R}^n$, $\vec{e}_i$ is the vector with its i^{th} component equal to one and the other $n-1$ components equal to zero.

The third condition $\det(A)$ must satisfy is:

$$\det(I_n) \;=\; \det\begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vdots \\ \vec{e}_n \end{bmatrix} \;=\; 1$$

The Weierstrass Definition

The third and final condition in the Weierstrass definition says that for any positive integer n, $\det(I_n) = 1$.

Recall that in $\mathbb{R}^n$, $\vec{e}_i$ is the vector with its i^{th} component equal to one and the other $n-1$ components equal to zero.

The third condition $\det(A)$ must satisfy is:

$$\det(I_n) \; = \; \det \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vdots \\ \vec{e}_n \end{bmatrix} \; = \; 1$$

In function notation,

$$f(\vec{e}_1, \ldots, \vec{e}_n) \; = \; 1$$

The Weierstrass Definition

In summary, for a square matrix A the determinant function $\det(A)$ has three properties:

- $\det(A)$ is linear in each row of A.
- Interchanging two rows changes the sign of $\det(A)$.
- The determinant of the identity matrix $\det(I)$ is 1.

The Weierstrass Definition

In summary, for a square matrix A the determinant function $\det(A)$ has three properties:

- $\det(A)$ is linear in each row of A.
- Interchanging two rows changes the sign of $\det(A)$.
- The determinant of the identity matrix $\det(I)$ is 1.

It can be shown that for each positive integer n, there is exactly one function

$$\det(A) : \mathbb{R}^{n\times n} \to \mathbb{R} \;=\; f(a_{11}, \ldots, a_{nn})$$

that has these three properties.

The Weierstrass Definition

The Weierstrass definition of the determinant is considered by many to be the most mathematically elegant of the various ways of defining $\det(A)$.

The Weierstrass Definition

The Weierstrass definition of the determinant is considered by many to be the most mathematically elegant of the various ways of defining $\det(A)$.

In advanced linear algebra courses, it is usually the method of choice for defining the determinant.

The Weierstrass Definition

By way of illustration, we will use the Weierstrass definition to give a proof of the following theorem:

Theorem: If A is a square matrix with two identical rows, then $\det(A) = 0$

The Weierstrass Definition

Proof: The Weierstrass definition of the determinant says that interchanging any two rows will reverse the sign of the determinant.

Let A^* be the matrix A with the identical rows interchanged. Then

$$\det(A^*) = -\det(A)$$

The Weierstrass Definition

Proof: The Weierstrass definition of the determinant says that interchanging any two rows will reverse the sign of the determinant.

Let A^* be the matrix A with the identical rows interchanged. Then

$$\det(A^*) \;=\; -\det(A)$$

Since the rows we interchanged are identical, $A^* = A$ and so

$$\det(A^*) \;=\; \det(A)$$

The Weierstrass Definition

Proof: The Weierstrass definition of the determinant says that interchanging any two rows will reverse the sign of the determinant.

Let A^* be the matrix A with the identical rows interchanged. Then

$$\det(A^*) \;=\; -\det(A)$$

Since the rows we interchanged are identical, $A^* = A$ and so

$$\det(A^*) \;=\; \det(A)$$

These two equations imply that $-\det(A) = \det(A)$, and this can only be true if

$$\det(A) = 0$$