The Laplace Expansion

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Minors

Definition: For an $n \times n$ matrix A, let A_{ij} be the $(n-1) \times (n-1)$ matrix obtained by deleting the i^{th} row and j^{th} column of A.

The determinant $\det(A_{ij})$ is called the ij^{th} **minor** of A.

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An $n \times n$ matrix A has n^2 minors, one for each element of A.

Minors

Example: Suppose the matrix A is

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 1 & 3 & 1 \\ 3 & 5 & 11 \end{bmatrix}$$

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For this matrix, A_{23} is obtained by removing the second row and third column of A,

$$\begin{bmatrix} 2 & 4 & \\ & & \\ 3 & 3 & \end{bmatrix} \quad \text{leaving} \quad A_{23} = \begin{bmatrix} 2 & 4 \\ 3 & 3 \end{bmatrix}$$

Minors

The 23^{th} minor of A is the determinant of A_{23},

$$\det(A_{23}) \;=\; \det\begin{bmatrix} 2 & 4 \\ 3 & 3 \end{bmatrix} \;=\; 2\cdot 3 - 3\cdot 4 \;=\; -6$$

The Laplace Expansion

Now associate a sign ($+$ or $-$) with each element of the $n \times n$ matrix A by the following rule: The sign associated with a_{ij}, the ij^{th} element of A, is

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For a 3×3 matrix, the pattern of signs is

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

The Laplace Expansion

The expression

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

is called the *Laplace expansion of* $\det(A)$ *down the* j^{th} *column* for $j = 1, \ldots, n$.

The Laplace Expansion

The expression

$$\det(A) \; = \; \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

is called the *Laplace expansion of* $\det(A)$ *across the* i^{th} *row* for $i = 1, \ldots, n$.

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is called the *Laplace expansion of* $\det(A)$ *across the* i^{th} *row* for $i = 1, \ldots, n$.

The Laplace expansion produces the same value for $\det(A)$ regardless of which row or column is chosen.

The Laplace Expansion

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The Laplace expansion provides a way to express the determinant of an $n \times n$ matrix in terms of the determinants of smaller $(n-1) \times (n-1)$ matrices.

This lends itself to recursive computational algorithms and proofs that use induction on the order of the minors.

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The Laplace expansion can be useful for computation when a matrix contains mostly zeros in one or more rows or columns.

The Laplace Expansion

The Bretscher text *defines* the determinant of an $n \times n$ matrix A as the Laplace expansion down the first column:

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The determinant of a 1×1 matrix

$$A = [a_{11}]$$

is defined to be a_{11}.

The Laplace Expansion

Example: Use the Laplace expansion to find the determinant of the matrix

$$A = \begin{bmatrix} 2 & 4 & 0 \\ 1 & 3 & 1 \\ 3 & 5 & 0 \end{bmatrix}$$

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We can choose any row or column for the Laplace expansion. The best choice is usually the one with the most zeros, so we will expand down the third column.

The Laplace Expansion

$$A = \begin{bmatrix} 2 & 4 & 0 \\ 1 & 3 & 1 \\ 3 & 5 & 0 \end{bmatrix}$$

The third column elements have $j = 3$, so the Laplace expansion is:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+3} a_{i3} \det(A_{i3})$$

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$$= (-1)^4 a_{13} \det(A_{13}) + (-1)^5 a_{23} \det(A_{23}) + (-1)^6 a_{33} \det(A_{33})$$

The Laplace Expansion

$$A = \begin{bmatrix} 2 & 4 & 0 \\ 1 & 3 & 1 \\ 3 & 5 & 0 \end{bmatrix}$$

$$\det(A) = 0 \cdot \det \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix} - 1 \cdot \det \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} + 0 \cdot \det \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$$

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$$\det(A) = -1(2 \cdot 5 - 4 \cdot 3) = -1(-2) = 2$$