Gauss-Jordan Elimination

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Consider the following system of linear equations:

$$\left|\begin{array}{rcrcrcr} x & + & y & + & z & = & 6 \\ 3x & + & 2y & - & 2z & = & 1 \\ -x & - & y & + & z & = & 0 \end{array}\right|$$

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The matrix

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 3 & 2 & -2 & 1 \\ -1 & -1 & 1 & 0 \end{bmatrix}$$

which contains only the numerical information in the system, is called the **augmented matrix**:

Systems of Linear Equations

It is easier to work with the augmented matrix (on the right) than the system itself because we aviod copying a lot of repetitive notation:

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We obtain the same solution whether we work with the equations or the augmented matrix.

The following matrix, which contains only the coefficients of the variables in the system, is called the **coefficient matrix**:

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

Row Reduction

Think of a pointer or cursor set to the leftmost element of the first row of the augmented matrix. The entry to which the cursor points will be called the *cursor entry*.

In the procedure, the row and column containing the cursor will be called the *cursor row* and *cursor column*, respectively.

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- **Step 4**: Move the cursor down one row, and to the right until it is pointing to the leftmost nonzero entry in the new cursor row. Now repeat from Step 1.

Row Reduction

We continue repeating steps 1-4 of the Row Reduction procedure until we run out of columns, at which point the process terminates.

Now the augmented matrix is in *reduced row-echelon form*, or *rref* for short, and if M is the original augmented matrix and E is the final augmented matrix, we write

$$E = \text{rref}(M)$$

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A matrix is said to be in *reduced row-echelon form* if the following conditions are true:

- a) If a row has any nonzero entries, the leftmost of them is 1, and is called the *leading* 1 or *pivot* in this row.
- b) If a column contains a leading 1 or pivot, then all other entries in that column are zero.
- c) If a row contains a leading 1 or pivot, then each row above it contains a leading 1 farther to the left.

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In summary, our procedure for solving a system of linear equations is:

- write down the linear system as an augmented matrix
- perform row reduction to reduce the augmented matrix to reduced row-echelon form
- determine if the system is consistent from the $rref$ (i.e., no $0 = 1$ equation in the $rref$)
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This procedure for solving a system of linear equations is called **Gauss-Jordan Elimination**.