Cramer's Rule

Gene Quinn

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We now introduce a method called **Cramer's rule** that gives the solution of a linear system in closed form.

Cramer's rule is mainly useful for obtaining theoretical results, because it is more expensive computationally than alternatives such as Gauss-Jordan.

Cramer's Rule

Cramer's Rule: Given a linear system

$$A\vec{x} = \vec{b}$$

where A is an **invertible** $n \times n$ matrix, the i^{th} component of the solution vector $\vec{x}$ is:

$$x_i = \frac{\det(A_{\vec{b},i})}{\det(A)}$$

where

$$A_{\vec{b},i}$$

is the matrix A with its i^{th} column replaced by $\vec{b}$.

Cramer's Rule

Example: Solve the following linear system $A\vec{x} = \vec{b}$ with Cramer's rule:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

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$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad A_{\vec{b},1} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \quad A_{\vec{b},2} = \begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix}$$

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and

$$\det(A) = 1 \qquad \det(A_{\vec{b},1}) = 1 \qquad \det(A_{\vec{b},2}) = 2$$

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The solution of the system is:

$$x_1 = \frac{\det(A_{\vec{b},1})}{\det(A)} = \frac{1}{1} = 1$$

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It is easily verified that

$$A\vec{x} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \vec{b}$$

The Classical Adjoint

Definition: The **classical adjoint** of an $n \times n$ matrix A is the matrix whose $(ij)^{th}$ entry is

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$$\text{adj}(A) = \begin{bmatrix} (-1)^2 A_{11} & (-1)^3 A_{21} & \cdots & (-1)^{n+1} A_{n1} \\ (-1)^3 A_{12} & (-1)^4 A_{22} & \cdots & (-1)^{n+2} A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+1} A_{1n} & (-1)^{n+2} A_{2n} & \cdots & (-1)n + nA_{nn} \end{bmatrix}$$

The Classical Adjoint

If an $n \times n$ matrix A is invertible, we can express A^{-1} in closed form as

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

The Classical Adjoint

Note that the $(ij)^{th}$ entry of adj(A) is the $(ji)^{th}$ minor of A,

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For example,

$$\text{adj}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$