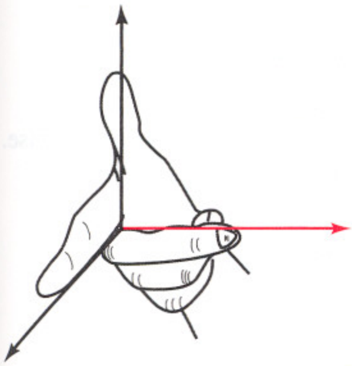
# APPENDIX A



# Vectors

Here we will provide a concise summary of basic facts on vectors. In Section 1.2, vectors are defined as matrices with only one column: $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$. The scalars $v_i$ are called the *components* of the vector.[1] The set of all vectors with $n$ components is denoted by $\mathbb{R}^n$.

You may be accustomed to a different notation for vectors. Writing the components in a column is the most convenient notation for linear algebra.

## Vector Algebra

**Definition A.1**

**Vector addition and scalar multiplication**

a. The sum of two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ is defined "componentwise":

$$\vec{v} + \vec{w} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix}$$

b. *Scalar multiplication* The product of a scalar $k$ and a vector $\vec{v}$ is defined componentwise as well:

$$k\vec{v} = k \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} kv_1 \\ kv_2 \\ \vdots \\ kv_n \end{bmatrix}$$

**EXAMPLE 1**

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 3 \end{bmatrix}$$ ■

[1] In vector and matrix algebra, the term "scalar" is synonymous with (real) number.

**EXAMPLE 2**

$$3\begin{bmatrix}1\\2\\0\\-1\end{bmatrix}=\begin{bmatrix}3\\6\\0\\-3\end{bmatrix}$$ ■

The *negative* or *opposite* of a vector $\vec{v}$ in $\mathbb{R}^n$ is defined as

$$-\vec{v}=(-1)\vec{v}\ .$$

The *difference* $\vec{v}-\vec{w}$ of two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ is defined componentwise. Alternatively, we can express the difference of two vectors as

$$\vec{v}-\vec{w}=\vec{v}+(-\vec{w})\ .$$

The vector in $\mathbb{R}^n$ that consists of $n$ zeros is called the *zero vector* in $\mathbb{R}^n$:

$$\vec{0}=\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}$$

**Fact A.2**

**Rules of vector algebra**

The following formulas hold for all vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ in $\mathbb{R}^n$ and for all scalars $c$ and $k$:

1. $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$. (Addition is *associative*.)
2. $\vec{v}+\vec{w}=\vec{w}+\vec{v}$. (Addition is *commutative*.)
3. $\vec{v}+\vec{0}=\vec{v}$.
4. For each $\vec{v}$ in $\mathbb{R}^n$, there is a unique $\vec{x}$ in $\mathbb{R}^n$ such that $\vec{v}+\vec{x}=0$, namely, $\vec{x}=-\vec{v}$.
5. $k(\vec{v}+\vec{w})=k\vec{v}+k\vec{w}$
6. $(c+k)\vec{v}=c\vec{v}+k\vec{v}$
7. $c(k\vec{v})=(ck)\vec{v}$
8. $1\vec{v}=\vec{v}$

These rules follow from the corresponding rules for scalars (commutativity, associativity, distributivity); for example:

$$\vec{v}+\vec{w}=\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}+\begin{bmatrix}w_1\\w_2\\\vdots\\w_n\end{bmatrix}=\begin{bmatrix}v_1+w_1\\v_2+w_2\\\vdots\\v_n+w_n\end{bmatrix}=\begin{bmatrix}w_1+v_1\\w_2+v_2\\\vdots\\w_n+v_n\end{bmatrix}$$

$$=\begin{bmatrix}w_1\\w_2\\\vdots\\w_n\end{bmatrix}+\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}=\vec{w}+\vec{v}.$$

## Geometrical Representation of Vectors

The *standard representation* of a vector

$$\vec{x}=\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

in the Cartesian coordinate plane is as an *arrow* (a directed line segment) connecting the origin to the point $(x_1, x_2)$, as shown in Figure 1.

Occasionally, it is helpful to translate (or shift) the vector in the plane (preserving its direction and length), so that it will connect some point $(a_1, a_2)$ to the point $(a_1 + x_1, a_2 + x_2)$. See Figure 2.

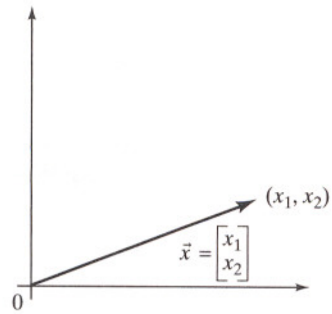



Figure 1

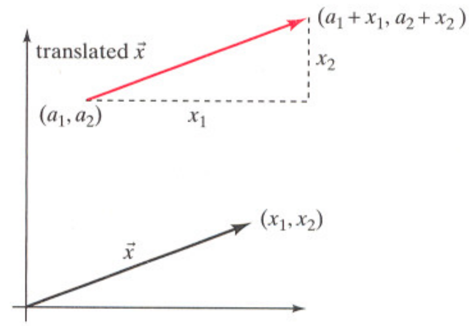



Figure 2

In this text, we consider the *standard representation* of vectors, unless we explicitly state that the vector has been translated.

A vector in $\mathbb{R}^2$ (in standard representation) is uniquely determined by its endpoint. Conversely, with each point in the plane we can associate its *position vector,* which connects the origin to the given point. See Figure 3.

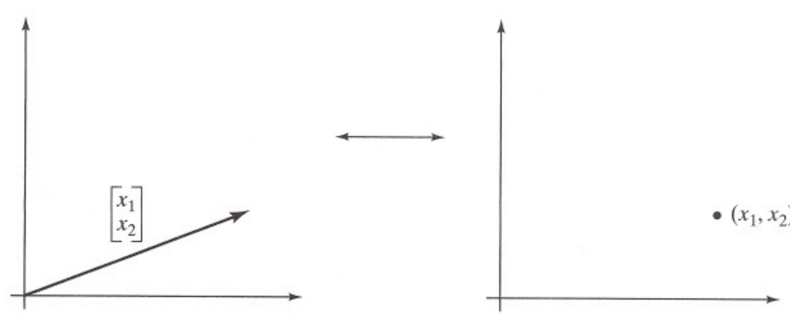



**Figure 3** The components of a vector in standard representation are the coordinates of its endpoint.

We need not clearly distinguish between a vector and its endpoint; we can identify them as long as we consistently use the standard representation of vectors.

For example, we will talk about "the vectors on a line $L$" when we really mean the vectors whose endpoints are on the line $L$ (in standard representation). Likewise, we can talk about "the vectors in a region $R$" in the plane. See Figure 4.

Adding vectors in $\mathbb{R}^2$ can be represented by means of a parallelogram, as shown in Figure 5.

If $k$ is a positive scalar, then $k\vec{v}$ is obtained by stretching the vector $\vec{v}$ by a factor of $k$, leaving its direction unchanged. If $k$ is negative, then the direction is reversed. See Figure 6.

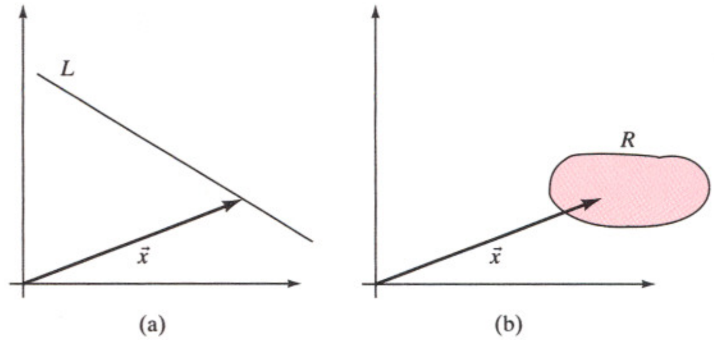



**Figure 4** (a) $\vec{x}$ is a vector on the line $L$. (b) $\vec{x}$ is a vector in the region $R$.

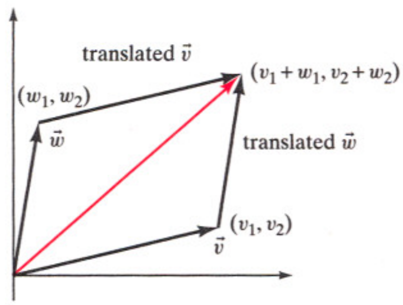



**Figure 5**

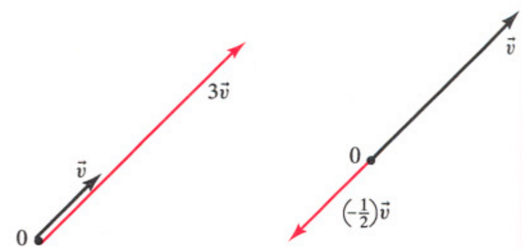



**Figure 6**

**Definition A.3** We say that two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ are *parallel* if one of them is a scalar multiple of the other.

**EXAMPLE 3** The vectors

$$\begin{bmatrix} 1 \\ 3 \\ 2 \\ -2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 \\ 9 \\ 6 \\ -6 \end{bmatrix}$$

are parallel, since

$$\begin{bmatrix} 3 \\ 9 \\ 6 \\ -6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 3 \\ 2 \\ -2 \end{bmatrix}.$$

■

**EXAMPLE 4** The vectors

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

are parallel, since

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

■

Let us briefly review Cartesian coordinates in *space*: If we choose an origin 0 and three mutually perpendicular coordinate axes through 0, we can describe any point in space by a triple of numbers, $(x_1, x_2, x_3)$. See Figure 7.

The standard representation of the vector

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

is the arrow connecting the origin to the point $(x_1, x_2, x_3)$, as shown in Figure 8.

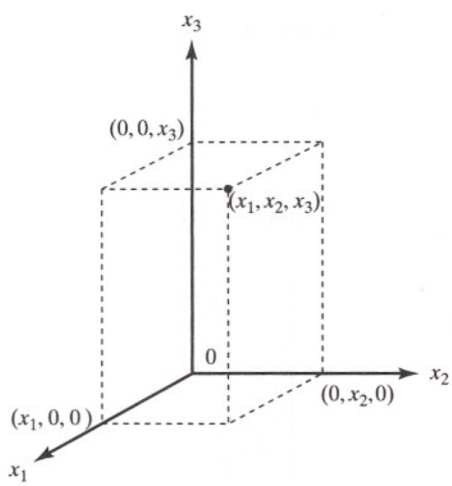



Figure 7

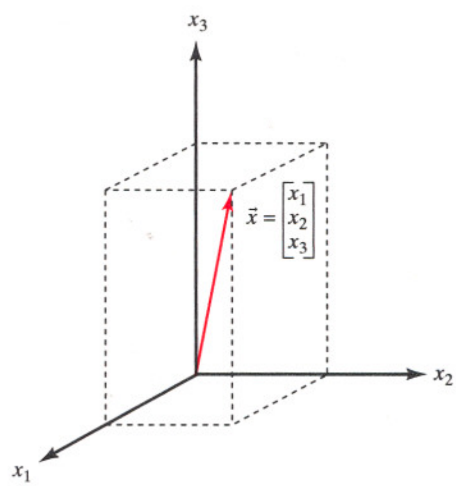



Figure 8

## Dot Product, Length, Orthogonality

**Definition A.4** Consider two vectors $\vec{v}$ and $\vec{w}$ with components $v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_n$, respectively. Here $\vec{v}$ and $\vec{w}$ may be column or row vectors, and they need not be of the same type (these conventions are convenient in linear algebra). The *dot product* of $\vec{v}$ and $\vec{w}$ is defined as

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

Note that the dot product of two vectors is a *scalar*.

**EXAMPLE 5**

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} = 1 \cdot 3 + 2 \cdot (-1) + 1 \cdot (-1) = 0.$$

■

**EXAMPLE 6**

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 0 \\ -1 \end{bmatrix} = 3 + 2 + 0 - 4 = 1.$$

■

**Fact A.5**

**Rules for dot products**

The following equations hold for all column or row vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ with $n$ components, and for all scalars $k$:

1. $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$.
2. $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$.
3. $(k\vec{v}) \cdot \vec{w} = k(\vec{v} \cdot \vec{w})$.
4. $\vec{v} \cdot \vec{v} > 0$ for all nonzero $\vec{v}$.

The verification of these rules is straightforward. Let us justify rule (d): since $\vec{v}$ is nonzero, at least one of the components $v_i$ is nonzero, so that $v_i^2$ is positive. Then

$$\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + \cdots + v_i^2 + \cdots + v_n^2$$

is positive as well.

Let us think about the *length* of a vector. The length of a vector

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

in $\mathbb{R}^2$ is $\sqrt{x_1^2 + x_2^2}$ by the Pythagorean theorem. See Figure 9.

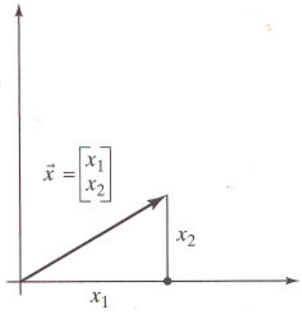



**Figure 9**

This length is often denoted by $\|\vec{x}\|$. Note that we have

$$\vec{x} \cdot \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + x_2^2 = \|\vec{x}\|^2;$$

therefore,

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}.$$

Verify that this formula holds for vectors $\vec{x}$ in $\mathbb{R}^3$ as well.

We can use this formula to *define* the length of a vector in $\mathbb{R}^n$:

**Definition A.6**

The *length* (or *norm*) $\|\vec{x}\|$ of a vector $\vec{x}$ in $\mathbb{R}^n$ is

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

**EXAMPLE 7** Find $\|\vec{x}\|$ for

$$\vec{x} = \begin{bmatrix} 7 \\ 1 \\ 7 \\ -1 \end{bmatrix}.$$

**Solution**

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{49 + 1 + 49 + 1} = 10.$$ ■

**Definition A.7** A vector $\vec{u}$ in $\mathbb{R}^n$ is called a *unit vector* if $\|\vec{u}\| = 1$; that is, the length of the vector $\vec{u}$ is 1.

Consider two perpendicular vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^2$, as shown in Figure 10.

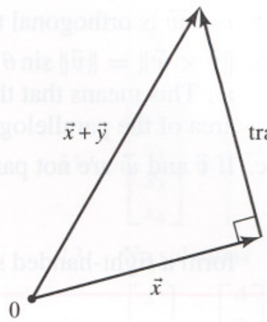



Figure 10

By the theorem of Pythagoras,

$$\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2,$$

or

$$(\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) = \vec{x} \cdot \vec{x} + \vec{y} \cdot \vec{y}.$$

By Fact A.5,

$$\vec{x} \cdot \vec{x} + 2(\vec{x} \cdot \vec{y}) + \vec{y} \cdot \vec{y} = \vec{x} \cdot \vec{x} + \vec{y} \cdot \vec{y},$$

or

$$\vec{x} \cdot \vec{y} = 0.$$

You can read these equations backward to show that $\vec{x} \cdot \vec{y} = 0$ if and only if $\vec{x}$ and $\vec{y}$ are perpendicular. This reasoning applies to vectors in $\mathbb{R}^3$ as well.

We can use this characterization to *define* perpendicular vectors in $\mathbb{R}^n$:

**Definition A.8** Two (row or column) vectors $\vec{v}$ and $\vec{w}$ are called *perpendicular* (or *orthogonal*) if $\vec{v} \cdot \vec{w} = 0$.

## Cross Product

Here we present the cross product for vectors in $\mathbb{R}^3$ only; for a generalization to $\mathbb{R}^n$, see Exercises 6.2.44 and 6.3.17.

In Chapter 6, we discuss the cross product in the context of linear algebra.

**Definition A.9**

The cross product of two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ is

$$\vec{v} \times \vec{w} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix}.$$

Unlike the dot product, the cross product $\vec{v} \times \vec{w}$ is a *vector* in $\mathbb{R}^3$.

**EXAMPLE 8**

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \times \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$$

∎

**Fact A.10**

**Geometric interpretation of the cross product**

Let $\vec{v} \times \vec{w}$ be the cross product of two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$. Then

a. $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$.

b. $\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \sin\theta \|\vec{w}\|$, where $\theta$ is the angle enclosed by the vectors $\vec{v}$ and $\vec{w}$. This means that the length of the vector $\vec{v} \times \vec{w}$ is numerically equal to the area of the parallelogram defined by $\vec{v}$ and $\vec{w}$. (See Figure 11a.)

c. If $\vec{v}$ and $\vec{w}$ are not parallel, then the vectors

$$\vec{v}, \ \vec{w}, \ \vec{v} \times \vec{w}$$

form a right-handed system. (See Figure 11b.)

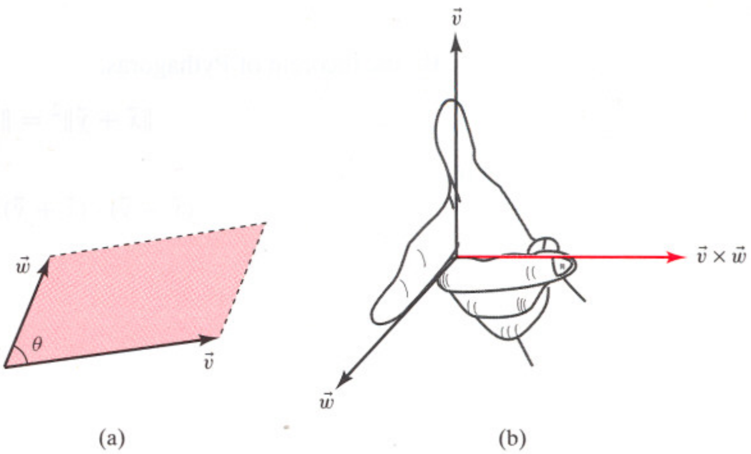



**Figure 11** (a) $\|\vec{v} \times \vec{w}\|$ is numerically equal to the shaded area. (b) A right-handed system.