MA251 Takehome Exam 2

Name:

1) If
$$T_1 : \mathbb{R}^m \to \mathbb{R}^k \quad \text{and} \quad T_2 : \mathbb{R}^k \to \mathbb{R}^n$$

are linear transformations with $T_1(\vec{x}) = B\vec{x},\ \vec{x} \in \mathbb{R}^m$ and $T_2(\vec{y}) = A\vec{y},\ \vec{y} \in \mathbb{R}^k$, then the composite transform
$$T_2(T_1(\vec{x})) : \mathbb{R}^m \to \mathbb{R}^n$$

has
$$T_2(T_1(\vec{x})) = AB\vec{x}$$

Show that $\ker(B) \subseteq \ker(AB)$. (hint: For sets C and D, proving $C \subseteq D$ is equivalent to proving that if $\vec{x} \in C$, then $\vec{x} \in D$.

2) If $T : \mathbb{R}^m \to \mathbb{R}^m$ has $T(\vec{x}) = A\vec{x}$ where
$$\begin{bmatrix} 2 & 2 & 2 & 2 \\ 1 & 2 & 0 & 4 \\ 3 & 2 & 4 & 0 \end{bmatrix}$$

a) Find bases for $\text{im}(A)$ and $\ker(A)$

b) Find the dimensions of $\text{im}(A)$ and $\ker(A)$

3) Find the matrix A of a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ whose image is a plane containing the points $(2, 4, 5)$, and $(1, 3, 2)$.

4) Characterization nine (*ix.*) of invertible matrices in Summary 3.3.9 on page 133 states that the columns of an $n \times n$ invertible matrix A are linearly independent.

Prove that this statement is true if we assume the first eight characterizations are true. (Hint: suppose they are linearly dependent and obtain a contradiction with one of the first eight characterizations).

5) Prove that if U and V are subspaces of $\mathbb{R}^n$ then their intersection $W = U \cap V$ is also a subspace of $\mathbb{R}^n$. (hint: Use the fact that $\vec{x} \in U \cap V$ if and only if $\vec{x} \in U$ and $\vec{x} \in V$).

6) Find two subspaces U and V of $\mathbb{R}^2$ whose union is *not* a subspace of $\mathbb{R}^2$ (Recall that $\vec{x} \in U \cup V$ if either $\vec{x} \in U$ or $\vec{x} \in V$ or both).

7) Determine which if any of the vectors

$\vec{v}_1 = (1, 1, 3, 1) \quad \vec{v}_2 = (2, 2, 1, 2) \quad \vec{v}_3 = (20, 20, 0, 20) \quad \vec{v}_4 = (-30, -30, -30, -30)$

are redundant.

8) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ has $T(\vec{x}) = A\vec{x}$. Suppose $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^3$ are linearly *dependent*. Prove that the vectors $T(\vec{x})$, $T(\vec{y})$, and $T(\vec{z})$ are also linearly dependent.

9) Prove that any set of two or more vectors in $\mathbb{R}^n$ that contains the zero vector is linearly dependent (Hint: show that the zero vector is redundant in any set with two or more vectors).

10) Let V be a subspace of $\mathbb{R}^n$. Define the **orthogonal compliment** of V, denoted by $V^\perp$, as the set of all vectors in $\mathbb{R}^n$ that are perpendicular to *every* vector in V. Recalling that two vectors are perpendicular if their dot product is zero, in set builder notation

$$V^\perp = \{\vec{u} : \vec{u} \in \mathbb{R}^n \quad \text{and} \quad \vec{u} \cdot \vec{v} = 0 \quad \text{for every} \quad \vec{v} \in V\}$$

Prove that $V^\perp$ is a subspace or $\mathbb{R}^n$.

11) Suppose V is a subspace of $\mathbb{R}^n$ and the vectors

$$\vec{v}_1,\ \vec{v}_2, \ldots, \vec{v}_m \qquad m \leq n$$

are a basis for V. If $\vec{x}$ is an element of V, show that there is only one way to write $\vec{x}$ as a linear combination of $\vec{v}_1, \ldots, \vec{v}_m$:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m$$

12) For what values, if any, of s, t, and u is the vector $\vec{v} = (1, 2, 1)$ in the kernel of

$$A = \begin{bmatrix} 1 & 2s & t \\ 0 & s & 2t \\ 1 & 1 & u \end{bmatrix} \quad ?$$

13) Find a basis for the span of the following set of vectors:

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 11 \\ 8 \\ 19 \\ 13 \end{bmatrix} \quad \begin{bmatrix} 8 \\ 8 \\ 16 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 2 \\ 5 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \\ 4 \\ 2 \end{bmatrix}$$

14) The *row space* of a matrix A is defined as the span of the row vectors that comprise A. That is, the row space is the set of all possible linear combinations of the rows of A.

Because all of the steps in transforming A to reduced row-echelon form involve linear operations on the rows of A, the row space of $rref(A)$ is the same as the row space of (A).

In particular, the nonzero rows of the $rref(A)$ form a basis for the row space of A.

Find a basis for the row space of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \\ 3 & 5 & 7 \end{bmatrix}$$

15) Use the definition of the reduced row-echelon form and the fact that the nonzero rows of $rref(A)$ form a basis for the row space of A to show that

$$\dim(\text{row space of } A) \;=\; \dim(\text{image of } A) \;=\; \text{rank of } A$$

16) Find bases for the image and kernel of the following matrix:

$$A \;=\; \begin{bmatrix} 11 & 8 & 10 \\ 8 & 8 & 16 \\ 2 & 2 & 4 \\ 1 & 2 & 6 \\ 3 & 2 & 2 \end{bmatrix}$$

17) Two vectors $\vec{v}_1$ and $\vec{v}_2$ are *perpendicular* if $\vec{v}_1 \cdot \vec{v}_2 = 0$. Suppose the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ have:

$$\vec{v}_i \cdot \vec{v}_j \;=\; \begin{cases} 0 & \text{if } \; i \neq j \\ \|\vec{v}_i\|^2 > 0 & \text{if } \; i = j \end{cases}$$

Show that the vectors in the set are linearly independent. Hint: for each i, $1 \leq i \leq n$, form the dot product of $\vec{v}_i$ with both sides of the equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n \;=\; \vec{0}$$

18) An $n \times n$ matrix is called *nilpotent* if $A^m = 0$ for some positive integer m (the zero on the right hand side is the $n \times n$ matrix consisting of all zeros).

Suppose A is a 3×3 matrix with $A^3 = 0$ and 3 is the smallest positive integer for which $A^m = 0$. For an arbitrary vector $\vec{v} \in \mathbb{R}^3$ chosen so that $A^2\vec{v} \neq \vec{0}$, show that the vectors

$$\vec{v},\ A\vec{v},\ A^2\vec{v}$$

are linearly independent. Hint: multiply both sides of the equation

$$c_0\vec{v} + c_1A\vec{v} + c_2A^2\vec{v} \;=\; \vec{0}$$

by A^2 to show that $c_0 = 0$, then by A to show that $c_1 = 0$, and so on.