Equally Likely Outcomes

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In fact this is one of the most common types of probability experiments.

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First determine the **number** of possible outcomes:

- Tossing a "fair" coin - 2 outcomes
- Rolling a balanced die - 6 outcomes
- Drawing a single card from a shuffled deck of 52 - 52 outcomes
- Spinning a roulette wheel - 38 outcomes

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We'll call this number N

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In the roll of a die, $N = 6$, so

$$p(1) = p(2) = p(3) = p(4) = p(5) = p(6) = \frac{1}{6}$$

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In the roulette "experiment", $N = 38$ because there are slots for 0, 00, and integers from 1 to 36. For each number

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For compound events, it's still fairly easy to compute the probability of the event.

We just have to add up the probabilities of the simple events that belong to the compound event.

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In an experiment with N equally likely outcomes, the probability of any event E is:

$$p(E) = \frac{N(E)}{N}$$

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The event "a number less than 5 is rolled" contains four simple events,

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This is how the probability of any event E in an experiment with equally likely outcomes is computed.

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As a result, computing probabilities of events in experiments with equally likely outcomes boils down to **counting**:

- Counting the total number of possible outcomes N
- Counting the number of simple events in an event $E, \quad N(E)$

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- Counting the total number of possible outcomes N
- Counting the number of simple events in an event $E, \quad N(E)$

The probability is always

$$p(E) = \frac{N(E)}{N}$$

Equally Likely Outcomes

We have three main tools for counting

- The product rule
- Permutations
- Combinations

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Recall that if I have to make a list of k elements with:

- n_1 choices for the first entry in the list
- n_2 choices for the second entry in the list (after the first choice is made)
- n_3 choices for the third entry in the list (after the first two choices)

and so on, the number of possible lists is $n_1 \cdot n_2 \cdot n_3 \cdots n_k$

Equally Likely Outcomes

If I have to select a list of k elements from a set of n, and order of selection matters, the number of possible lists is given by the **permutation**

$$P_{k,n} = \frac{n!}{(n-k)!}$$

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$$P_{k,n} = \frac{n!}{(n-k)!}$$

If I have to select a list of k elements from a set of n, and order of selection is irrelevant, the number of possible lists is given by the **combination**

$$C_{k,n} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

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- Royal straight flush (A,K,Q,J,10 in the same suit)
- Straight flush (Five cards in sequence in the same suit)
- Four of a kind (Four cards of the same denomination)
- Flush (Five cards in the same suit)
- Straight (Five consecutive denominations)
- Full House (Three of one denomination, two of another)
- Three of a Kind (Three cards of one denomination)
- Two Pair (Two pairs of cards in the same denomination)
- One Pair (two cards in the same denomination)

Equally Likely Outcomes

If we think of each of these as an event, the probability of each will be

$$p(E) = \frac{N(E)}{N}$$

where $N(E)$ is the number of possible hands for this event, and N is the total number of possible hands.

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If we think of each hand as a list of 5 cards chosen randomly from a deck of 52, with order not important, the number of hands is then the combination:

$$C_{5,52} = \frac{52!}{5!(47)!} = \binom{52}{5}$$

The easiest way to compute this is to use a spreadsheet formula:

=COMBIN(52,5)

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If we do this, we find there are

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We can now find the probability of any hand if we can determine $N(E)$, the number of ways that hand can occur.

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Example: Find the probability of a royal straight flush.

First, we have to find $N(E)$, the number of possible different hands that qualify as a royal straight flush.

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The only thing we can choose is the suit.

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Now the probability of a royal straight flush is:

$$p(E) = \frac{N(E)}{N} = \frac{4}{2598960}$$

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Example: Find the probability of four of a kind.

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Assuming we have removed 4 cards of one denomination, there remain 48 possible choices for the fifth card.

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Assuming we have removed 4 cards of one denomination, there remain 48 possible choices for the fifth card.

By the product rule, $N(E) = 13 \cdot 48 = 624$, so the probability of four of a kind is:

$$p(E) = \frac{N(E)}{N} = \frac{624}{2598960}$$

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Example: Find the probability of three of a kind.

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Think of first choosing the denomination for the three cards, which we can do in 13 ways.

Once we have made that choice, we will pick three of the four cards of that denomination to be in the hand.

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Since order is not important, this amounts to making an unordered list of 3 from a group of 4, $C_{3,4} = 4$

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Once we have made that choice of denomination, we will pick the suit.

Since order is not important, this amounts to making an unordered list of 1 from a group of 4, $C_{1,4} = 4$

Equally Likely Outcomes

Now for the fifth card. We have to pick a denomination different from that of the first three cards, or we would have four of a kind. We have to pick something different that the fourth card, because otherwise we have a full house (three of one denomination, two of another). So, we have 11 choices this time.

There are no restrictions on the suit of the fourth card, so we have four choices again.

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By the product rule, the total number of hands with three of a kind is the product of the number of choices at each step:

$$N(E) = 13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{1} \cdot 11 \cdot \binom{4}{1} = 109,824$$

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The probability of three of a kind is then: