Continuous Random Variables

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Measurements of continuous quantities are usually represented as continuous random variables:

- temperature
- salinity
- pH
- elapsed time

Probability Density Functions

If X is a continuous random variable, the *probability density function* (pdf) of X is a function $f(x)$ with the property that, for any $a \leq b$,

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx$$

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Any function with the following properties is a valid pdf:

$$f(x) \geq 0 \quad \text{for all } x \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)\, dx = 1$$

Example 1

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$$f(x) = 2x \quad 0 \leq x \leq 1$$

Find

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By definition,

$$P\left(\frac{1}{4} \le X \le \frac{3}{4}\right) = \int_{1/4}^{3/4} 2x\ dx = x^2\Big|_{1/4}^{3/4} = \frac{9}{16} - \frac{1}{16} = \frac{1}{2}$$

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Uniform Distribution

A continuous random variable X has a **uniform distribution** on the interval $[A, B]$ if its pdf is:

$$f(x; A, B) = \frac{1}{B - A} \quad A \leq x \leq B$$

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Cumulative Distribution Functions

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From the fundamental theorem of calculus,

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_{-\infty}^{x} f(y)\, dy = f(x)$$

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The above inequality implies that the probability that X equals any single value is zero:

$$P(a \leq X \leq a) = F(a) - F(a) = 0$$

This takes a bit of getting used to.

Cumulative Distribution Functions

The cdf can be used to determine **percentiles** of a distribution. The $100p^{th}$ percentile $\nu(p)$ of the distribution of X satisfies

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The **median** of the distribution of X is $\nu(.5)$ and satisfies

$$P(X \leq \nu(0.5)) = F(\nu(0.5)) = 0.5$$

Expected Value

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The expected value of a function $h(x)$ random variable X is defined as

$$\mu_{h(x)} = E(h(X)) = \int_{-\infty}^{\infty} h(x) \cdot f(x)\, dx$$

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The standard deviation (SD) of X is

$$\sigma_x = \sqrt{V(x)}$$