Sullivan Section 6.2

Gene Quinn

Binomial Probability Experiments

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- The probability of success is the same for each trial

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If the probability of success is $1/2$, the binomial experiment is equivalent to a series of coin tosses.

Binomial Probability Experiments

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- The probability of success on each trial is denoted by p.
- The probability of failure on each trial is $1 - p$.
- The total number of successes in n independent trials is denoted by X.

Computing Binomial Probabilities

The following structure known as **Pascal's triangle** is useful for computing binomial probabilities when n is fairly small ($n < 10$).

									1								
$n = 1$								1		1							
$n = 2$							1		2		1						
$n = 3$						1		3		3		1					
$n = 4$					1		4		6		4		1				
$n = 5$				1		5		10		10		5		1			
$n = 6$			1		6		15		20		15		6		1		
$n = 7$		1		7		21		35		35		21		7		1	

Computing Binomial Probabilities

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The entries in successive rows of Pascal's triangle are the sum of the two closest entries in the previous row.

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The sum of the entries in the row corresponding to n trials is always 2^n.

This represents the number of possible sequences of n letters where each one has to be either S or F.

Computing Binomial Probabilities

There are always $n+1$ entries in the row corresponding to n trials.

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- The fourth entry is the number of sequences having 3 $S's$ and $n-3$ $F's$

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- The next to last entry is the number of sequences having $n - 1\ S's$ and $1\ F$ (the next to last entry is always n)
- The last entry is the number of sequences having $n\ S's$ and $0\ F's$ (the last entry is always 1)

Binomial Probabilities when $p = 0.5$

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If we identify "heads" with "success", the experiment corrsponds tossing a fair coin n times.

In this case, the probability of obtaining 0, 1, 2, etc. heads in n tosses is the corresponding entry in Pascal's table, divided by the sum of the row (2^n).

Binomial Probabilities when $p \neq 0.5$

When success and failure are *not* equally likely, we need to use the following modified procedure to calculate the probabilities.

The number of trials n determines which row of Pascal's triangle is used.

Binomial Probabilities when $p \neq 0.5$

Suppose the probability of success on each trial is p.

We compute the probabilities associated with each value of X,
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- Continue in this fashion. The $n+1^{st}$ entry is multiplied by $(1-p)^n$

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For convenience, we define $0!$ to be 1.

Computing Binomial Probabilities

Definition: The number of **combinations** of n objects taken r at a time is denoted by either

$$_nC_r \qquad \text{or} \qquad \binom{n}{r}$$

and is defined to be:

$$\frac{n!}{r!(n-r)!}$$

Computing Binomial Probabilities

Example: Find the number of **combinations** of 4 objects taken 2 at a time.

That is, find

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By definition,

$$ \binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{(2 \cdot 1) \cdot (2 \cdot 1)} = \frac{24}{(2)(2)} = 6 $$

Computing Binomial Probabilities in G

The general formula for computing the probability of k successes in a binomial experiment with n trials when the probability of success on each trial is p is:

$$P(k \text{ successes}) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n$$

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or, equivalently,

$$P(k \text{ successes}) =_n C_k p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n$$

Mean of a Binomial Random Variable

Suppose the criteria for a binomial probability experiment are met.

The possible outcomes of the experiment, and the probabilities associated with each outcome are completely determined by two numbers:

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Furthermore, we know that for $k = 0, 1, \ldots, 6$, the probability that exactly k successes are obtained is given by the formula:

$$P(X = k) \quad = \quad {}_6C_k \cdot p^k (1 - p)^{n-k}$$

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- The probability of 6 successes is ${}_6C_6 \cdot (0.6)^6(0.4)^0$

Means and Standard Deviations

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$$\sigma_X = \sqrt{n \cdot p \cdot (1 - p)}$$

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The standard deviation σ_X is given by

$$\sigma_X \;=\; \sqrt{n \cdot p \cdot (1 - p)} \;=\; \sqrt{100 \cdot 0.6 \cdot 0.4} \;=\; 4.90$$

Means, Standard Deviations, and the E

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How large is a "reasonably large" value of n? It depends on p.

A commonly used rule of thumb states that the binomial distribution will be approximately bell shaped provided that

$$n \geq \frac{10}{p \cdot (1-p)}$$

Means, Standard Deviations, and the E

Earlier we found that for a binomial experiment with 100 trials each having a probability of 0.6 of success, the mean and standard deviation were:

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$n = 100$ is more than adequate to satisfy the rule of thumb stating that n should be greater than or equal to $10/(p \cdot (1 - p))$, so the empirical rule tells us that:

- approximately 68% of the time X will fall in the range 55.1 to 64.9
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