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In other cases, the data values may represent laboratory measurements or samples of instrument readings.

In these situations we can use numerical techniques known as *approximate integration*

Approximate Integration

The three techniques we will study,

- The Midpoint Rule
- The Trapezoidal Rule
- Simpson's Rule

are slightly modified versions of the Riemann sums we used to find the area under curves.

Approximate Integration

The Midpoint Rule: evaluate f at the midpoint of each interval.

$$\int_a^b f(x)\,dx \;\approx\; M_n \;=\; \Delta x\left[f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)\right]$$

$$\Delta x = \frac{b-a}{n} \quad \text{and} \quad \overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$$

$\overline{x}_i$ is the midpoint of the interval $[x_{i-1}, x_i]$.

Approximate Integration

The Trapezoidal Rule: evaluate f at each end of the interval and average the result.

$$\int_a^b f(x)\,dx \;\approx\; T_n$$

$$T_n \;=\; \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

$$\Delta x = \frac{b-a}{n}$$

Approximate Integration

Simpson's Rule:

$$\int_a^b f(x)\,dx \;\approx\; S_n$$

$$S_n \;=\; \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots \right.$$
$$\left. \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

$$\Delta x = \frac{b-a}{n} \quad n \text{ must be even}$$

Error of Approximation

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Define the *error terms* for M_n, T_n, and S_n as:

$$E_M = \int_a^b f(x)dx - M_n \quad E_T = \int_a^b f(x)dx - T_n$$

and

$$E_S = \int_a^b f(x)dx - S_n$$

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Since we usually don't know the exact value of the integral, we cannot compute these errors exactly.

However, we can determine upper bounds for them. This is the worst case scenario.

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This means the Trapezoidal rule is exact for a linear function $f(x) = ax + b$

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This means for example that Simpson's rule is exact for a *cubic* polynomial.

Question 1

What is the maximum possible absolute error if the Midpoint rule is used to approximate

$$\int_0^2 \sin^2 x \, dx$$

using $n = 40$ points?

1. $\frac{2^3}{24 \cdot 40^2}$
2. $\frac{2^3}{12 \cdot 40^2}$
3. $\frac{1}{24 \cdot 40^2}$
4. $\frac{2^3}{24 \cdot 40}$
5. $\frac{2^2}{24 \cdot 40^2}$
6. none of the above

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Question 2

What is the maximum possible absolute error if Simpson's rule is used to approximate

$$\int_0^2 \sin^2 x \, dx$$

using $n = 40$ points?

1. $\frac{2^4}{180 \cdot 40^4}$
2. $\frac{2^4}{180 \cdot 20^4}$
3. $\frac{1}{180 \cdot 40^4}$
4. $\frac{2^4}{180 \cdot 40^3}$
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