Trigonometric Identities

It is possible to fill a book with trigonometric identities.

However, by far the most important is:

$$\cos^2 x + \sin^2 x = 1$$

Trigonometric Identities

It is possible to fill a book with trigonometric identities.

However, by far the most important is:

$$\cos^2 x + \sin^2 x = 1$$

This gives rise to the expressions

$$\cos^2 x = 1 - \sin^2 x \quad \text{and} \quad \sin^2 x = 1 - \cos^2 x$$

Trigonometric Identities

It is possible to fill a book with trigonometric identities.

However, by far the most important is:

$$\cos^2 x + \sin^2 x = 1$$

This gives rise to the expressions

$$\cos^2 x = 1 - \sin^2 x \quad \text{and} \quad \sin^2 x = 1 - \cos^2 x$$

From these we get

$$\cos x = \sqrt{1 - \sin^2 x} \quad \text{and} \quad \sin x = \sqrt{1 - \cos^2 x}$$

Trigonometric Identities

$$\cos x = \sqrt{1 - \sin^2 x} \quad \text{and} \quad \sin x = \sqrt{1 - \cos^2 x}$$

The above expressions can be used to turn any expression involving $\cos x$ into an equivalent in terms of $\sin x$, and vice-versa.

Trigonometric Identities

$$\cos x = \sqrt{1 - \sin^2 x} \quad \text{and} \quad \sin x = \sqrt{1 - \cos^2 x}$$

The above expressions can be used to turn any expression involving $\cos x$ into an equivalent in terms of $\sin x$, and vice-versa.

One reason you might need to do this is to evaluate an integral like

$$\int \sin^3 x$$

Trigonometric Identities

Using the identity $\sin^2 x = 1 - \cos^2 x$, we can write the integral as

$$\int \sin^3 x \; = \; \int (1 - \cos^2 x) \sin x \; dx$$

Trigonometric Identities

Using the identity $\sin^2 x = 1 - \cos^2 x$, we can write the integral as

$$\int \sin^3 x \;=\; \int (1 - \cos^2 x) \sin x \; dx$$

If we let $u = \cos x$, then

$$\frac{du}{dx} = -\sin x \quad \text{so} \quad -\,du = \sin x \; dx$$

Trigonometric Identities

Using the identity $\sin^2 x = 1 - \cos^2 x$, we can write the integral as

$$\int \sin^3 x \;=\; \int (1 - \cos^2 x) \sin x \; dx$$

If we let $u = \cos x$, then

$$\frac{du}{dx} = -\sin x \quad \text{so} \quad -du = \sin x \; dx$$

Now on substitution the integral becomes

$$\int \sin^3 dx = \int (1 - \cos^2 x) \sin x \; dx = -\int (1 - u^2) \; du$$

Trigonometric Identities

The result is:

$$\int \sin^3 dx \;=\; -\int (1-u^2)\, du \;=\; \frac{u^3}{3} - u$$

$$=\; \frac{\cos^3 x}{3} - \cos x + C$$

Trigonometric Identities

In general, this technique is useful for integrals of the form

$$\int \sin^m x \cos^n x \, dx$$

Trigonometric Identities

In general, this technique is useful for integrals of the form

$$\int \sin^m x \cos^n x \, dx$$

There are three special cases to consider, depending on whether m and n are even or odd.

Trigonometric Identities

$$\int \sin^m x \cos^n x \; dx$$

Case 1: m is odd, that is, $m = 2k + 1$ for some natural number k

Trigonometric Identities

$$\int \sin^m x \cos^n x \; dx$$

Case 1: m is odd, that is, $m = 2k + 1$ for some natural number k

In this case write the integral as

$$\int \sin^{2k+1} x \cos^n x \; dx \; = \; \int \sin^{2k} x \cos^n x \;\; \sin x \; dx$$

Trigonometric Identities

$$\int \sin^m x \cos^n x \; dx$$

Case 1: m is odd, that is, $m = 2k + 1$ for some natural number k

In this case write the integral as

$$\int \sin^{2k+1} x \cos^n x \; dx \; = \; \int \sin^{2k} x \cos^n x \; \sin x \; dx$$

Now substitute

$$(1 - \cos^2 x)^k \quad \text{for} \quad \sin^{2k}$$

Trigonometric Identities

The result is:

$$\int (1 - \cos^2 x)^k \cos^n x \ \sin x \ dx$$

Trigonometric Identities

The result is:

$$\int (1 - \cos^2 x)^k \cos^n x \ \sin x \, dx$$

Now let

$$u = \cos x \quad \text{so} \quad \frac{du}{dx} = -\sin x \ \text{ and } \ -du = \sin x \, dx$$

Trigonometric Identities

The result is:

$$\int (1 - \cos^2 x)^k \cos^n x \ \sin x \, dx$$

Now let

$$u = \cos x \quad \text{so} \quad \frac{du}{dx} = -\sin x \ \text{ and } \ -du = \sin x \, dx$$

On substitution the integral becomes

$$\int \sin^m x \cos^n x \, dx \ = \ -\int (1 - u^2)^k u^n \, du$$

Trigonometric Identities

The result is:

$$\int (1 - \cos^2 x)^k \cos^n x \ \sin x \ dx$$

Now let

$$u = \cos x \quad \text{so} \quad \frac{du}{dx} = -\sin x \ \text{ and } \ -du = \sin x \ dx$$

On substitution the integral becomes

$$\int \sin^m x \cos^n x \ dx \ = \ -\int (1 - u^2)^k u^n \ du$$

This can be either expanded and integrated as a polynomial, or integrated by parts.

Trigonometric Identities

$$\int \sin^m x \cos^n x \, dx$$

Case 2: n is odd, that is, $n = 2k + 1$ for some natural number k

Trigonometric Identities

$$\int \sin^m x \cos^n x \; dx$$

Case 2: n is odd, that is, $n = 2k + 1$ for some natural number k

In this case write the integral as

$$\int \sin^m x \cos^{2k+1} x \; dx \; = \; \int \sin^m x \cos^{2k} x \;\; \cos x \; dx$$

Trigonometric Identities

$$\int \sin^m x \cos^n x \; dx$$

Case 2: n is odd, that is, $n = 2k + 1$ for some natural number k

In this case write the integral as

$$\int \sin^m x \cos^{2k+1} x \; dx \; = \; \int \sin^m x \cos^{2k} x \;\; \cos x \; dx$$

Now substitute

$$(1 - \sin^2 x)^k \quad \text{for} \quad \cos^{2k}$$

Trigonometric Identities

The result is:

$$\int \sin^m x(1 - \sin^2 x)^k \cos x \; dx$$

Trigonometric Identities

The result is:

$$\int \sin^m x(1 - \sin^2 x)^k \cos x \, dx$$

Now let

$$u = \sin x \quad \text{so} \quad \frac{du}{dx} = \cos x \ \text{ and } \ du = \cos x \, dx$$

Trigonometric Identities

The result is:

$$\int \sin^m x(1 - \sin^2 x)^k \cos x \, dx$$

Now let

$$u = \sin x \quad \text{so} \quad \frac{du}{dx} = \cos x \ \text{ and } \ du = \cos x \, dx$$

On substitution the integral becomes

$$\int \sin^m x \cos^n x \, dx \ = \ -\int u^m (1 - u^2)^k \, du$$

Trigonometric Identities

The result is:

$$\int \sin^m x(1 - \sin^2 x)^k \cos x \, dx$$

Now let

$$u = \sin x \quad \text{so} \quad \frac{du}{dx} = \cos x \;\text{ and }\; du = \cos x \, dx$$

On substitution the integral becomes

$$\int \sin^m x \cos^n x \, dx \;=\; -\int u^m (1 - u^2)^k \, du$$

This can be either expanded and integrated as a polynomial, or integrated by parts.

Trigonometric Identities

$$\int \sin^m x \cos^n x \, dx$$

Case 3: m and n are both even: $m = 2j$ and $n = 2k$

Trigonometric Identities

$$\int \sin^m x \cos^n x \, dx$$

Case 3: m and n are both even: $m = 2j$ and $n = 2k$

In this case write the integral as

$$\int \sin^{2j} x \cos^{2k} x \, dx$$

Trigonometric Identities

$$\int \sin^m x \cos^n x \, dx$$

Case 3: m and n are both even: $m = 2j$ and $n = 2k$

In this case write the integral as

$$\int \sin^{2j} x \cos^{2k} x \, dx$$

Now substitute

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \quad \text{and} \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

Trigonometric Identities

The result is:

$$\int \frac{1}{2}(1 - \cos 2x)^j \frac{1}{2}(1 + \cos 2x)^k \, dx$$

Trigonometric Identities

The result is:

$$\int \frac{1}{2}(1 - \cos 2x)^j \frac{1}{2}(1 + \cos 2x)^k \; dx$$

This can be either expanded and integrated as a polynomial, or integrated by parts.