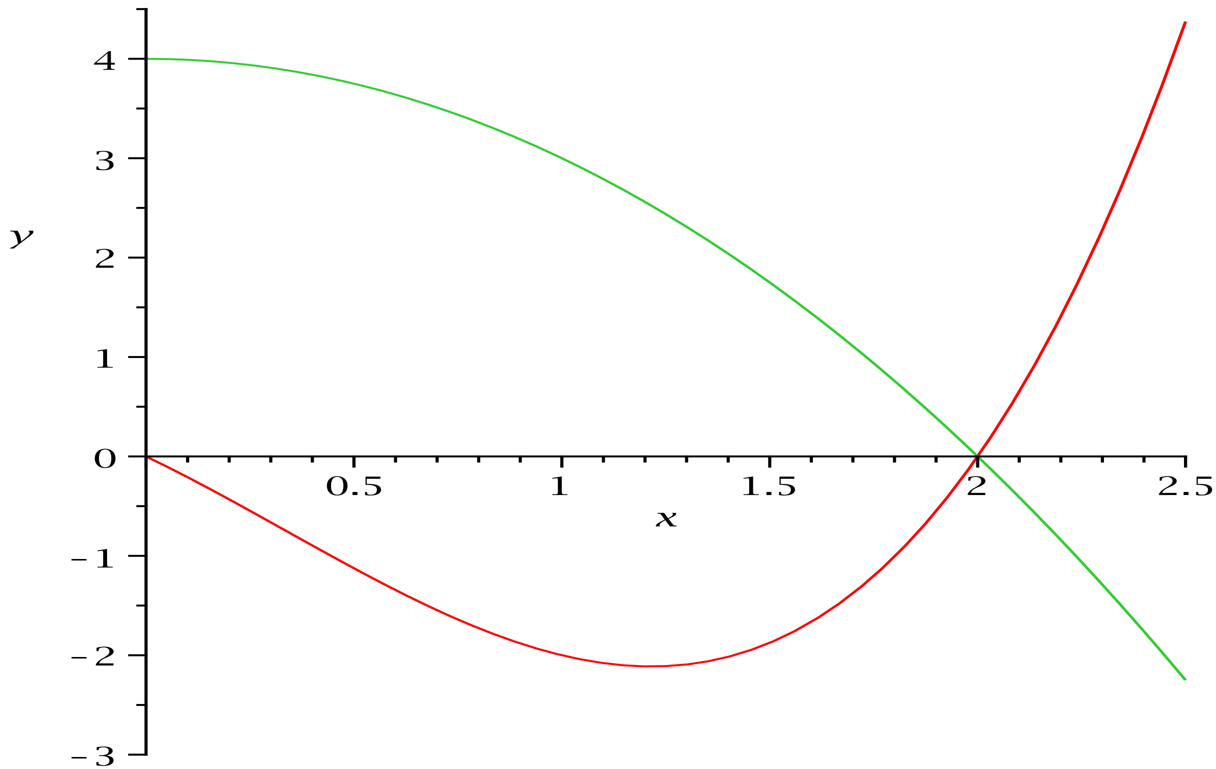
Areas Between Curves

Suppose we are interested in the area between the curves

$$f(x) = 4 - x^2 \quad \text{and} \quad g(x) = x^3 - x^2 - 2x$$



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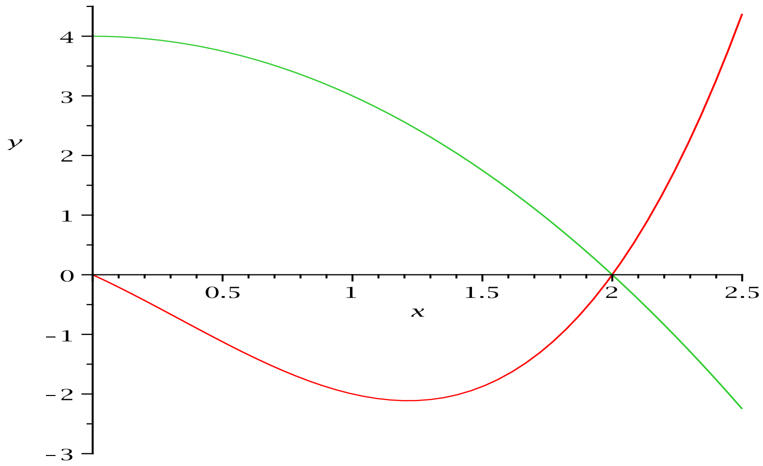

We need to be careful in this case because f is not always larger than g.

If our interval of integration is $[0, 2]$, $f \geq g$, so we can just use

$$A = \int_0^2 (f(x) - g(x))\ dx$$

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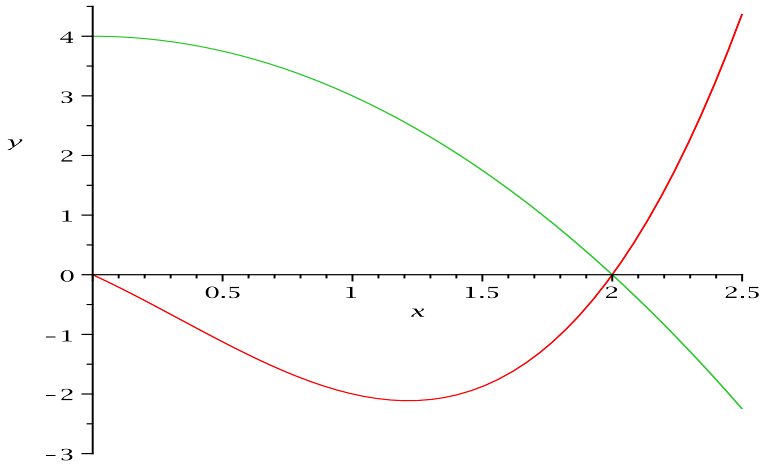

If our interval of integration is, say $[0, 2.5]$, we have to use the formula

$$A = \int_0^{2.5} |f(x) - g(x)| \ dx$$

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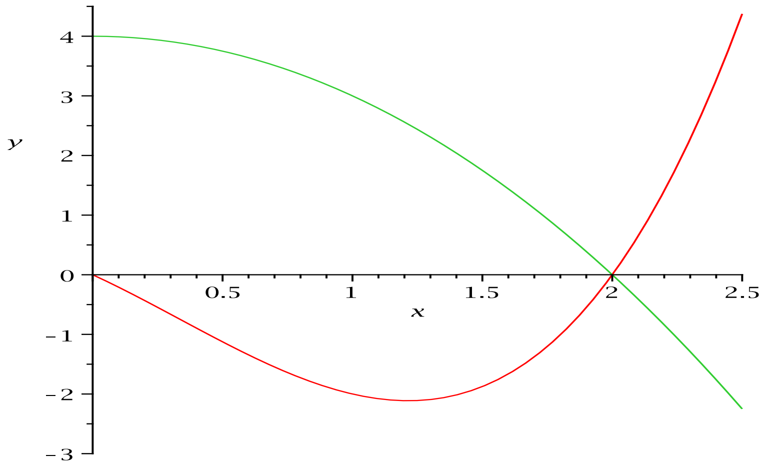

If our interval of integration is, say $[0, 2.5]$, we have to use the formula

$$A = \int_0^{2.5} |f(x) - g(x)| \; dx$$

The reason is that $f \geq g$ on $[0, 2]$, but $g \geq f$ on $[2, 2.5]$

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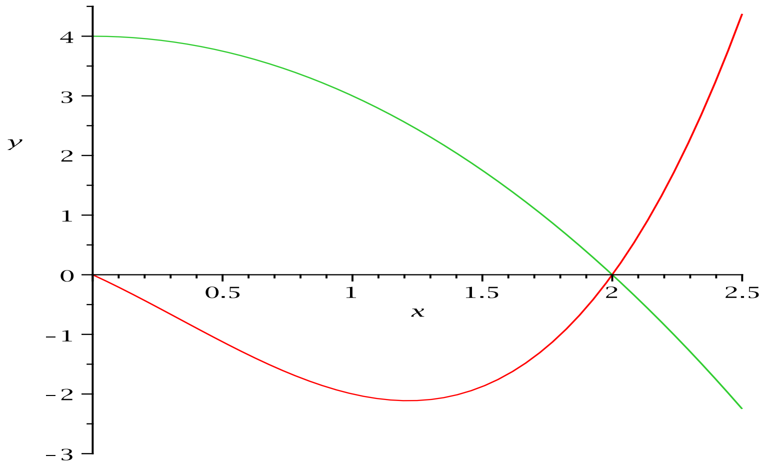

So we have to split the interval into two parts, and write

$$A = \int_0^{2.5} |f(x) - g(x)| \ dx$$

as

$$A = \int_0^{2} (f(x) - g(x)) \ dx + \int_2^{2.5} (g(x) - f(x)) \ dx$$

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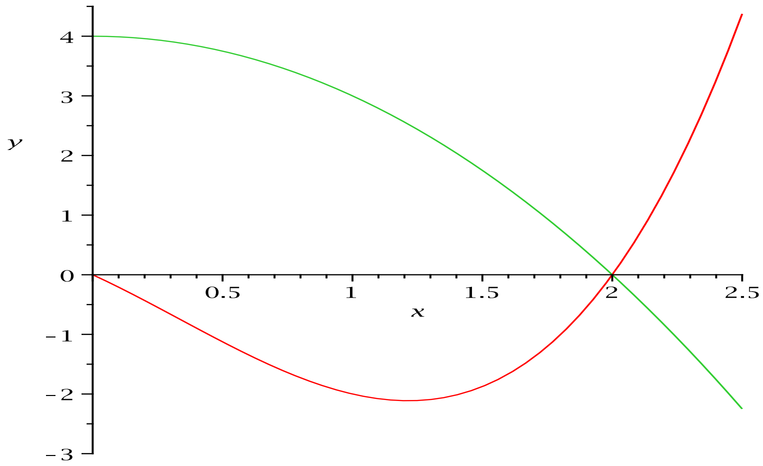

As a practical matter, we might also consider that since

$$\int_a^b (f(x) - g(x))\ dx = -\int_a^b (g(x) - f(x))\ dx$$

we can avoid two different integrations by using

$$A = \int_0^2 (f(x) - g(x))\ dx - \int_2^{2.5} (f(x) - g(x))\ dx$$

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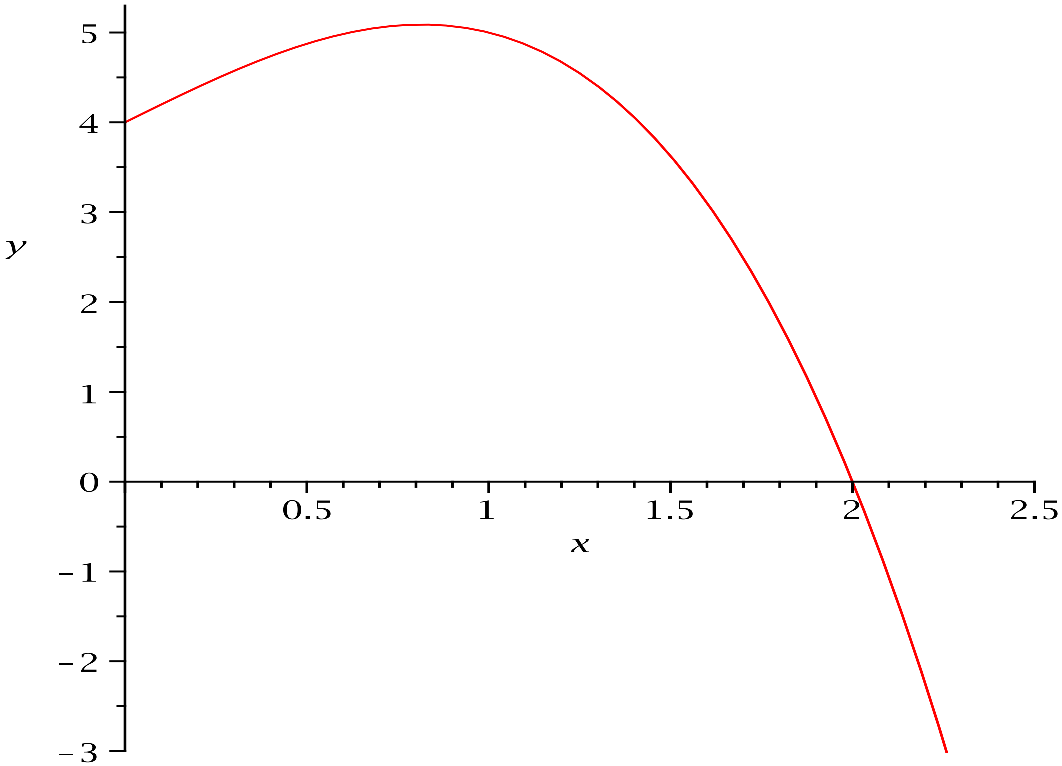
One technique for discovering this is to graph the function $(f-g)(x)$ on the interval of interest. In this case, we get



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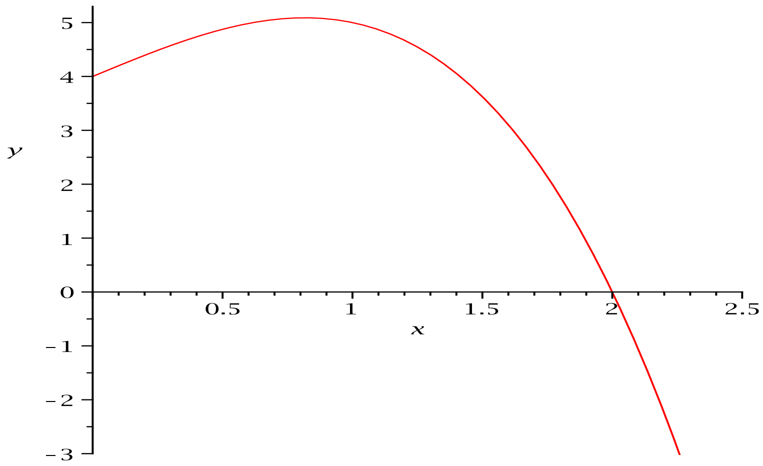

If the graph is always *on or above* the x-axis in the entire range of integration, we don't need to split the integral.

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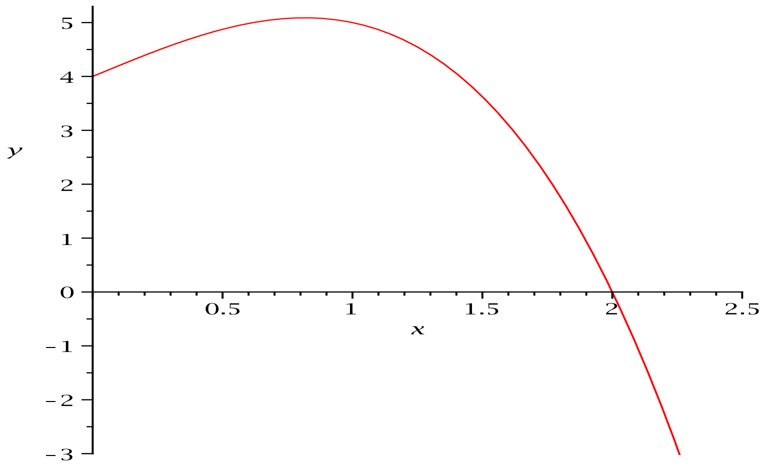

If the graph is always *on or above* the x-axis in the entire range of integration, we don't need to split the integral.

If the graph has values *both above and below* the x-axis in the range of integration, then we will have to split the integral.

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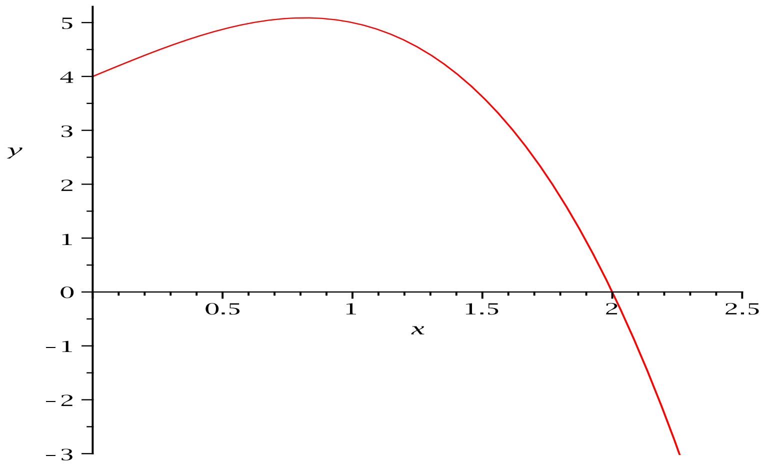

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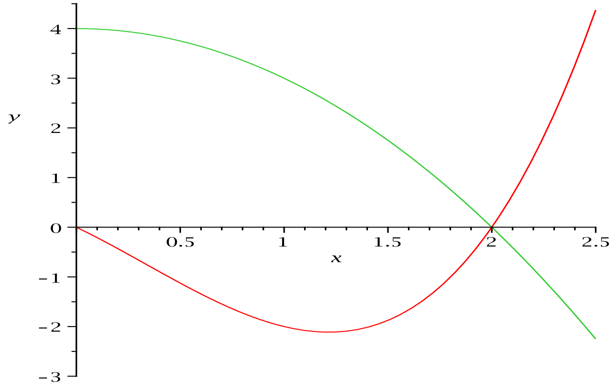
If the graph has values *both above and below* the x-axis in the range of integration, then we will have to split the integral.

If the graph is always *on or below* the x-axis in the range of integration, we do not have to split the integral, but we need to integrate $(g - f)(x)$ instead of $(f - g)(x)$

Areas Between Curves

Example 1: Find the area between the curves $f(x) = 4 - x^2$ and $g(x) = x^3 - x^2 - 2x$ between $x = 0$ and $x = 2$.



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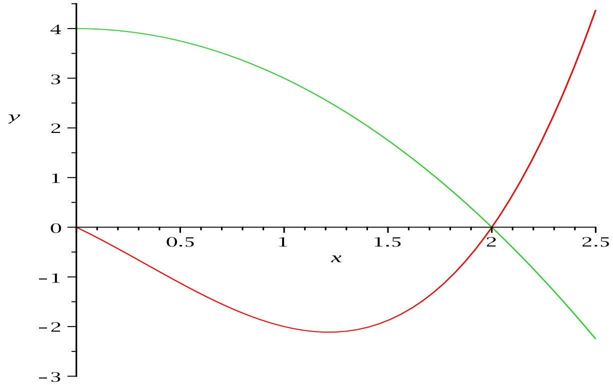

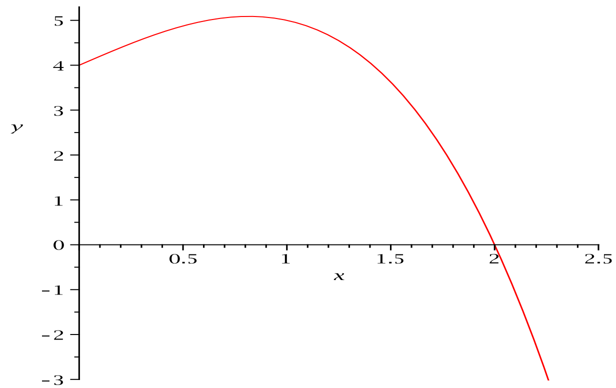

Plot $f(x) - g(x)$ on this interval and note that $f(x) \geq g(x)$ on $[0, 2]$.

Areas Between Curves

$$f(x) - g(x) = (4 - x^2) - (x^3 - x^2 - 2x) = -x^3 + 2x + 4$$

is nonnegative on $[0, 2]$, so we can simply integrate $(f - g)$ over this interval.

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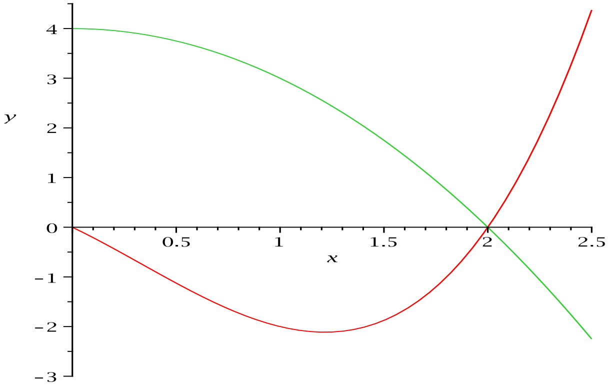
$$= \left[-\frac{16}{4} + 4 + 8\right] - \left[-\frac{0}{4} + 0^2 + 0\right] = 8$$

Areas Between Curves

Example 2: Find the area between the curves $f(x) = 4 - x^2$ and $g(x) = x^3 - x^2 - 2x$ between $x = 0$ and $x = 3$.



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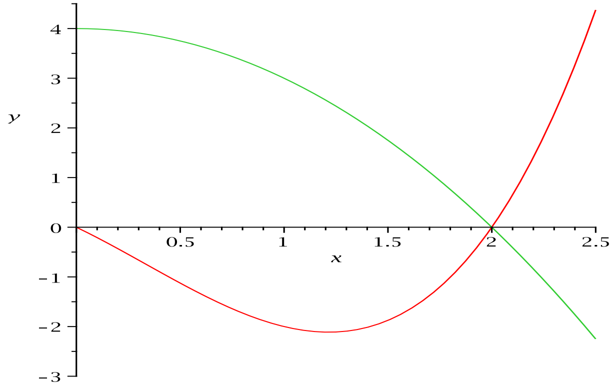

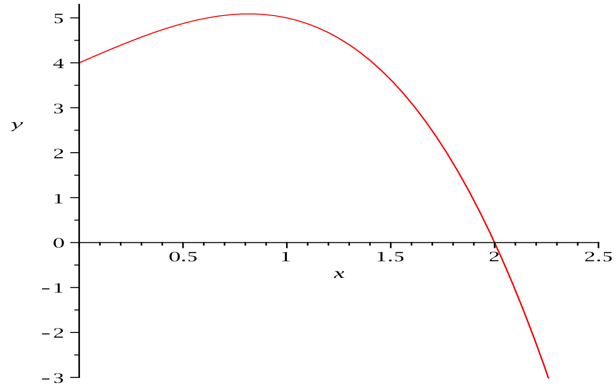

Plot $f(x) - g(x)$ on this interval and note that $f(x) \geq g(x)$ on $[0, 2]$, $g(x) \geq f(x)$ on $[2, 3]$.

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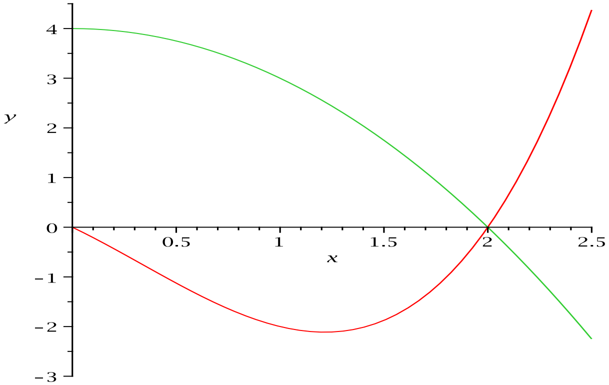

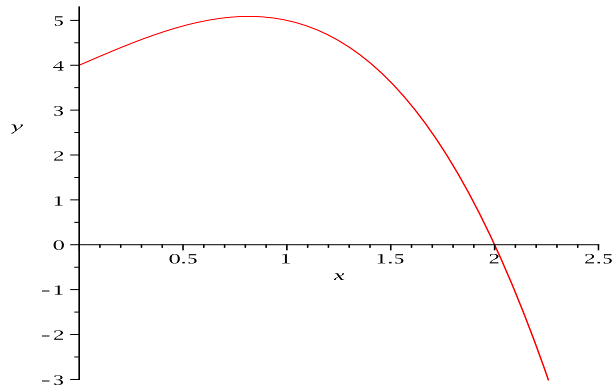

Plot $f(x) - g(x)$ on this interval and note that $f(x) \geq g(x)$ on $[0, 2]$, $g(x) \geq f(x)$ on $[2, 3]$. The sign of $(f - g)(x)$ changes at $x = 2$.

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which means we have to split the integral into segments where the sign of $(f-g)(x)$ does not change.
In this case the intervals are:

- $[0, 2]$ where $f(x) \geq g(x)$
- $[2, 3]$ where $g(x) \geq f(x)$

so

$$A = \int_0^2 [(4-x^2)-(x^3-x^2-2x)]dx + \int_2^3 [(x^3-x^2-2x)-(x^2-4)]dx$$

Areas Between Curves

By collecting terms,

$$A = \int_0^2 [(4 - x^2) - (x^3 - x^2 - 2x)]dx + \int_2^3 [(x^3 - x^2 - 2x) - (x^2 - 4)]dx$$

can be written as

$$A = \int_0^2 (-x^3 + 2x + 4)dx + \int_2^3 (x^3 - 2x - 4)dx$$

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After integration this is

$$\left[-\frac{x^4}{4} + x^2 + 4x\right]_0^2 + \left[\frac{x^4}{4} - x^2 - 4x\right]_2^3$$

Areas Between Curves

Evaluating at the endpoints we have

$$\left[-\frac{16}{4}+4+8\right]-\left[-\frac{0}{4}+0^2+0\right]$$

$$+\left[\frac{81}{4}-9-12\right]-\left[\frac{16}{4}-4-8\right]$$

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