The Substitution Rule - Definite Integrals

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The Substitution Rule - Definite Integrals

If:

- g' is continuous on $[a, b]$
- f is continuous on the range of $u = g(x)$

then

$$\int_a^b f\left(g(x)\right)\, g'(x)\, dx \;=\; \int_{g(a)}^{g(b)} f(u)\, du$$

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Example: Evaluate

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Solution: Let $f(u) = \sqrt{u}$ and $g(x) = u = 1 + x^2$. Then:

$$\frac{du}{dx} = 2x$$

or, in differential form,

$$du = 2xdx \quad \Rightarrow \quad dx = \frac{du}{2x}$$

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Now substituting of $du/2x$ for dx, we have

$$\int_{g(0)}^{g(4)} 2x\sqrt{u}\,\frac{du}{2x} \quad = \quad \int_1^{17} \sqrt{u}\,du \;=\; \frac{2}{3}u^{3/2}\Big|_1^{17}$$

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The final answer is

$$\frac{2}{3}\left(17^{\frac{3}{2}} - 1\right)$$

Note that the limits of integration changed when dx was replaced by du.

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An alternative to replacing the limits of integration,

$$a \longrightarrow g(a), \quad b \longrightarrow g(b)$$

is the following:

- Once the integrand is in the form $f(u)\,du$, find an antiderivative $F(u)$.
- Substitute $g(x)$ for u in the antiderivative, and use the original limits.

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Returning to the previous example, we would find an antiderivative $F(u)$

$$\int \sqrt{u}\,du \;=\; \frac{2}{3}u^{\frac{3}{2}} + C$$

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Now substitute for $u = g(x) = 1 + x^2$, and use the original limits of integration, 0 to 4:

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$$\frac{2}{3}(1+x^2)^{\frac{3}{2}}\Big|_0^4 = \frac{2}{3}\left(17^{\frac{3}{2}} - 1\right)$$

This is the same answer we obtained by evaluating

$$\frac{2}{3}u^{\frac{3}{2}}\Big|_1^{17} = \frac{2}{3}\left(17^{\frac{3}{2}} - 1\right)$$

with the modified limits $u = g(0) = 1$ and $u = g(4) = 17$.