The Substitution Rule

Gene Quinn

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- The range of g is some interval I
- f is continuous on I

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The **substitution rule** states that

$$\int f(g(x)) \cdot g'(x)dx = \int f(u)du$$

The Substitution Rule

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Namely, it allows us handle functions that are **compositions**, such as

$$\int \cos 3x \, dx$$

The following are the steps in applying the substitution rule:

The Substitution Rule

- identify f and g

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- write $u = g(x)$

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- find the differential dx in terms of du
- substitute u for $g(x)$ in the integrand
- substitute the expression involving du for dx

Example 1

Find

$$\int x \cdot \sin(x^2) dx$$

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The fist step is to identify f and g.

We want to choose f and g so that the integrand looks like this:

$$f(g(x)) \cdot g'(x)\,dx$$

Example 1

For

$$\int x \cdot \sin(x^2)\,dx$$

possible choices would be

$$f(u) = \sin u \quad \text{and} \quad g(x) = x^2$$

Example 1

With these choices,

$$f(g(x)) = \sin x^2 \quad \text{and} \quad g'(x) = 2x$$

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$$f(g(x)) = \sin x^2 \quad \text{and} \quad g'(x) = 2x$$

and, combining these,

$$f(g(x)) \cdot g'(x) \quad = \quad \sin(x^2) \cdot 2x$$

Example 1

Note that the expression,

$$\sin(x^2) \cdot 2x$$

is identical to our integrand

$$\sin(x^2) \cdot x$$

up to a constant.

Example 1

As it turns out, the constant doesn't really matter.

Using the property of integrals that states

$$\int c \cdot f(x)dx \quad = \quad c \cdot \int f(x)dx$$

we can always move it outside the integral sign.

Substitution Rule

- identify f and g
- write $u = g(x)$
- find du/dx
- find the differential dx in terms of du
- substitute u for $g(x)$ in the integrand
- substitute the expression involving du for dx

Example 1

For

$$\int x \cdot \sin(x^2)\, dx$$

Step 1. Choose:

- $f(u) = \sin u$
- $g(x) = x^2$

Example 1

Step 2. Let

$$u = g(x) = x^2$$

Example 1

Step 2. Let

$$u = g(x) = x^2$$

Step 3. Find

$$\frac{du}{dx} = 2x$$

Example 1

Step 4. Find dx in terms of du:

$$\frac{du}{dx} = 2x \quad \text{so} \quad du = 2x\,dx$$

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Step 4. Find dx in terms of du:

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Then, solving for dx,

$$dx = \frac{1}{2x}du$$

Example 1

Step 5. Substitute

$$u = g(x) = x^2$$

for $g(x)$ in the integrand:

$$\int x \cdot \sin(x^2)\,dx \quad = \quad \int x \cdot \sin(u)\,dx$$

Example 1

Step 6. Substitute

$$dx = \frac{1}{2x}\,du$$

for dx in the integrand:

$$\int x \cdot \sin(u)\,dx \quad = \quad \int x \cdot \sin(u) \cdot \frac{1}{2x}\,du$$

Example 1

Now simplify the integrand

The x's should cancel.

$$\int x \cdot \sin(u) \cdot \frac{1}{2x}\, du \quad = \quad \int \sin(u) \cdot \frac{x}{2x}\, du$$

Example 1

Now simplify the integrand

The x's should cancel.

$$\int x \cdot \sin(u) \cdot \frac{1}{2x}\,du \quad = \quad \int \sin(u) \cdot \frac{x}{2x}\,du$$

which becomes

$$\int \sin(u) \cdot \frac{1}{2}\,du \quad = \quad \frac{1}{2}\int \sin(u)\,du$$

Example 1

Now we find an antiderivative of $\sin(u)$,

$$\int \sin(u)\,du = -\cos(u) + C$$

Example 2

Finally, our answer is

$$-\frac{1}{2}\cdot\cos(u)+C \quad = \quad -\frac{1}{2}\cos(x^2)+C$$

Example 1

So

$$\int x \cdot \sin(x^2)\, dx = -\frac{1}{2}\cos(x^2) + C$$

Example 2

Find

$$\int \sqrt{3-5x}\,dx$$

Example 2

Step 1: Identify f and g

$$\int \sqrt{3-5x}\,dx$$

Example 2

Step 1: Identify f and g

$$\int \sqrt{3 - 5x}\, dx$$

$$f(u) = \sqrt{u} \quad , \quad g(x) = 3 - 5x$$

Example 2

Step 2: Write $u = g(x)$

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Step 2: Write $u = g(x)$

$$u = g(x) = 3 - 5x$$

Example 2

Step 3: Find du/dx

Example 2

Step 3: Find du/dx

$$u = 3 - 5x \quad , \quad \frac{du}{dx} = -5$$

Example 2

Step 4: Find dx in terms of du

Example 2

Step 4: Find dx in terms of du

$$du = -5\,dx \quad , \quad dx = -\frac{1}{5}\,du$$

Example 2

Step 5: Substitute u for $g(x)$ in the integrand.

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$$\int \sqrt{3 - 5x}\,dx \quad = \quad \int \sqrt{u}\,dx$$

Example 2

Step 6: Substitute the expression for dx in terms of du

Example 2

Step 6: Substitute the expression for dx in terms of du

$$dx = -\frac{1}{5}\,du$$

so

$$\int \sqrt{u}\,dx = \int \sqrt{u}\left(-\frac{1}{5}\right)\,du$$

Example 2

The final answer is

$$\int \sqrt{u}\left(-\frac{1}{5}\right)\,du = -\frac{1}{5}\int \sqrt{u}\,du$$

so using the power rule to find an antiderivative of $\sqrt{u}$ we have

$$-\frac{1}{5}\int \sqrt{u}\,du = -\frac{1}{5}\cdot\frac{2}{3}\cdot u^{3/2} + C$$

Example 2

In terms of the original variable x, this is

$$\int \sqrt{3-5x}\,dx \quad = \quad -\frac{2}{15}\cdot(3-5x)^{3/2}+C$$

Example 3

Find

$$\int x \cdot \exp(3x^2)\, dx$$

Example 3

Find

$$\int x \cdot \exp(3x^2)\,dx$$

Answer:

$$\frac{1}{6}\exp(3x^2) + C$$

Example 4

Find

$$\int \sin^4 x \cos x \, dx$$

Example 4

Find

$$\int \sin^4 x \cos x \, dx$$

Answer:

$$\frac{1}{5}(\sin x)^5 + C$$

Example 5

Find

$$\int \frac{\exp(x)}{1+\exp(x)}\,dx$$

Example 5

Find

$$\int \frac{\exp(x)}{1+\exp(x)}\,dx$$

Answer:

$$\ln(1+\exp(x))+C$$

Definite Integrals

For definite integrals,

$$\int_a^b f(g(x))g'(x)dx$$

there is only one change:

The limits of integration become $g(a)$ and $g(b)$:

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Definite Integrals

If Example 5 had been a definite integral,

$$\int_0^1 \frac{\exp(x)}{1+\exp(x)}\,dx$$

$g(x) = 1 + e^x$ and the answer would have been

$$\int_0^1 \frac{\exp(x)}{1+\exp(x)}\,dx = \ln(1+e^x)]_1^{1+e}$$