Riemann Sums

Gene Quinn

Definition of Riemann Sum

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Note that R_n is a Riemann sum with x_i^* chosen to be the **right** endpoint of the i^{th} interval.

Likewise L_n is a Riemann sum with x_i^* chosen to be the **left** endpoint of the i^{th} interval.

Riemann Sum Example

Example: Riemann sum evaluation. If:

- $f(x) = x^2$
- $a = 2$
- $b = 4$

calculate

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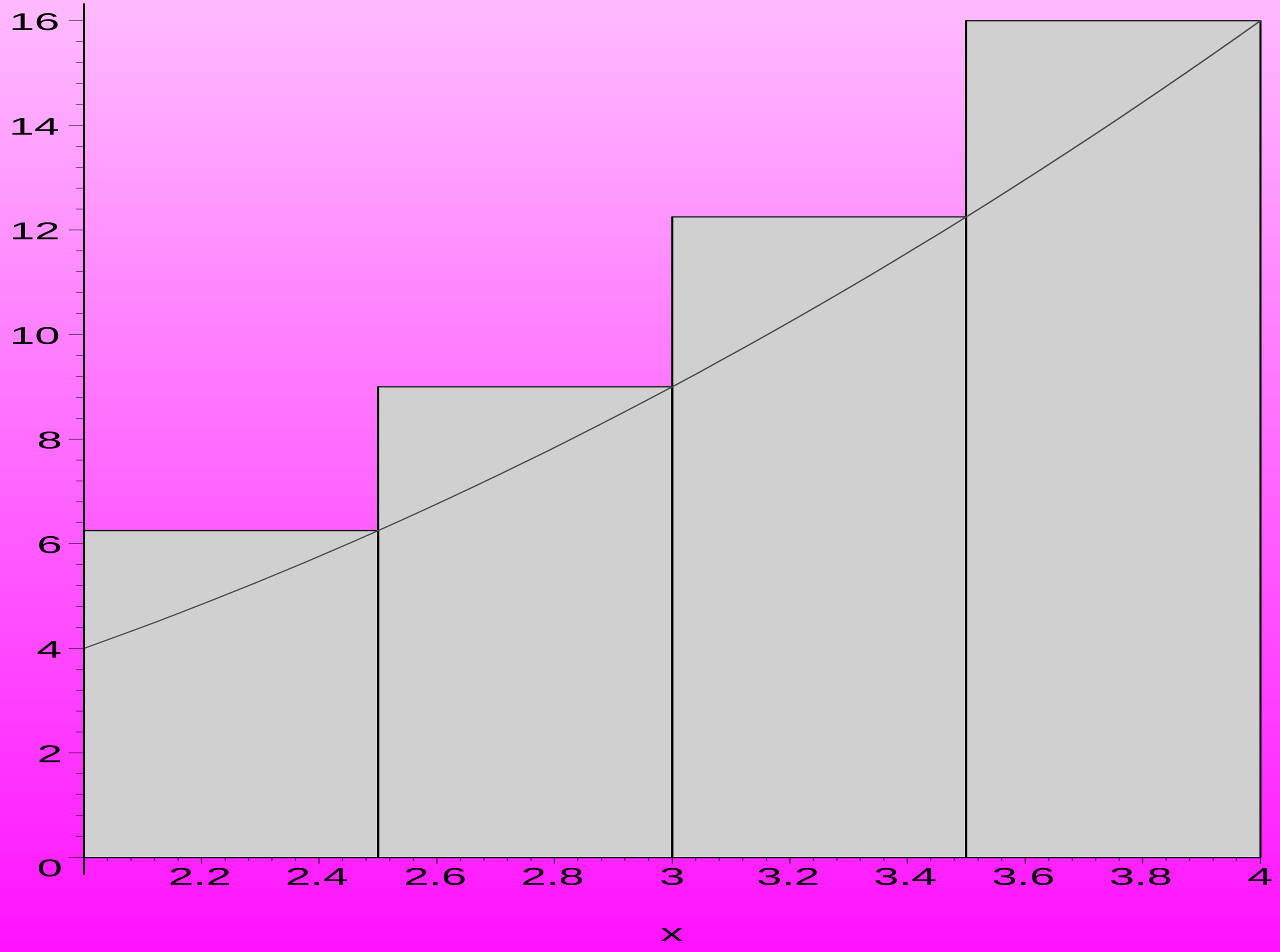
Solution: For R_4, $n = 4$.

First calculate Δx as:

$$\Delta x = \frac{b-a}{n} = \frac{4-2}{4} = 0.5$$

Riemann Sum Example

The picture in this case is:



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So, our Riemann sum R_4 has

$$x_i^* = x_i = a + i \cdot \Delta x = a + i, \quad i = 1, 2, 3, 4$$

$$\begin{aligned} x_1^* &= 2 + 1 * 0.5 &= 2.5 \\ x_2^* &= 2 + 2 * 0.5 &= 3 \\ x_3^* &= 2 + 3 * 0.5 &= 3.5 \\ x_4^* &= 2 + 4 * 0.5 &= 4 \end{aligned}$$

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$$R_4 = \quad 3.125 \quad + \quad 4.5 \quad + \quad 6.125 \quad + \quad 8$$

$$R_4 = 21.75$$

Riemann Sums

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If we call the right endpoint of the i^{th} subinterval x_i, then we have essentially defined a set of values

$$\{x_0 = a, x_1, x_2, \ldots, x_n = b\}$$

that partition the interval $[a, b]$ into n subintervals

$$\{[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, b]\}$$

Riemann Sums

Now choose a set of n *sample points*

$$\{x_1^*, x_2^*, \dots, x_n^*\}$$

such that x_i^* is in the i^{th} subinterval, that is,

$$x_{i-1} \le x_i^* \le x_i, \quad , i = 1, 2, \dots, n$$

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Sums of the form

$$\sum_{i=1}^{n} f(x_i^*) \Delta x$$

are called *Riemann sums*.

Riemann Sums - Special Cases

Here are a couple of special cases of Riemann sums for particular choices of the x_i^*.

If x_i^* is chosen to be the right endpoint of the i^{th} interval, then

$$x_i^* = x_i$$

and the Riemann sum is

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In this case, $x_i = a + i \cdot \Delta x$ so the Riemann sum can be written as

$$\sum_{i=1}^{n} f(a + i \cdot \Delta x)\Delta x = \sum_{i=1}^{n} f\left[a + i \cdot \left(\frac{b-a}{n}\right)\right]\left(\frac{b-a}{n}\right)$$

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If x_i^* is chosen to be the left endpoint of the i^{th} interval, then

$$x_i^* = x_{i-1}$$

and the Riemann sum is

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(recall that we defined $x_0 = a$)

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$$\sum_{i=1}^{n} f[a + (i-1)\cdot\Delta x]\Delta x = \sum_{i=1}^{n} f\left[a + (i-1)\cdot\left(\frac{b-a}{n}\right)\right]\left(\frac{b-a}{n}\right)$$

Riemann Sums - Special Cases

Finally, we may choose x_i^* to be the midpoint of the i^{th} interval. In this case, we denote x_i^* by $\overline{x}_i$:

$$x_i^* = \overline{x}_i = \frac{1}{2}\left(x_{i-1} + x_i\right)$$

and the Riemann sum is

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and the Riemann sum is

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In this case, $\overline{x}_i = a + (i - \frac{1}{2}) \cdot \Delta x$ so the Riemann sum can be written as

$$\sum_{i=1}^{n} f\left[a + \left(i - \frac{1}{2}\right) \cdot \Delta x\right] \Delta x = \sum_{i=1}^{n} f\left[a + \left(i - \frac{1}{2}\right) \cdot \left(\frac{b-a}{n}\right)\right] \left(\frac{b-a}{n}\right)$$

Applications of Riemann Sums

As we saw in section 5.1, sums of the form

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In many applications, the total change can be represented as the limit of a Riemann sum

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

as the number of subintervals increases without bound.