Properties of Definite Integrals

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Properties of Definite Integrals

In the following slides, we assume that

$$[a, b]$$

is an interval and that the functions

$$f(x) \quad \text{and} \quad g(x)$$

are both continuous on $[a, b]$.

Definite Integral of a Constant Function

Property 1:

If the function f is constant on an interval $[a, b]$, that is,

$$f(x) = c \quad a \leq x \leq b$$

then

$$\int_a^b f(x)dx = c \cdot (b - a)$$

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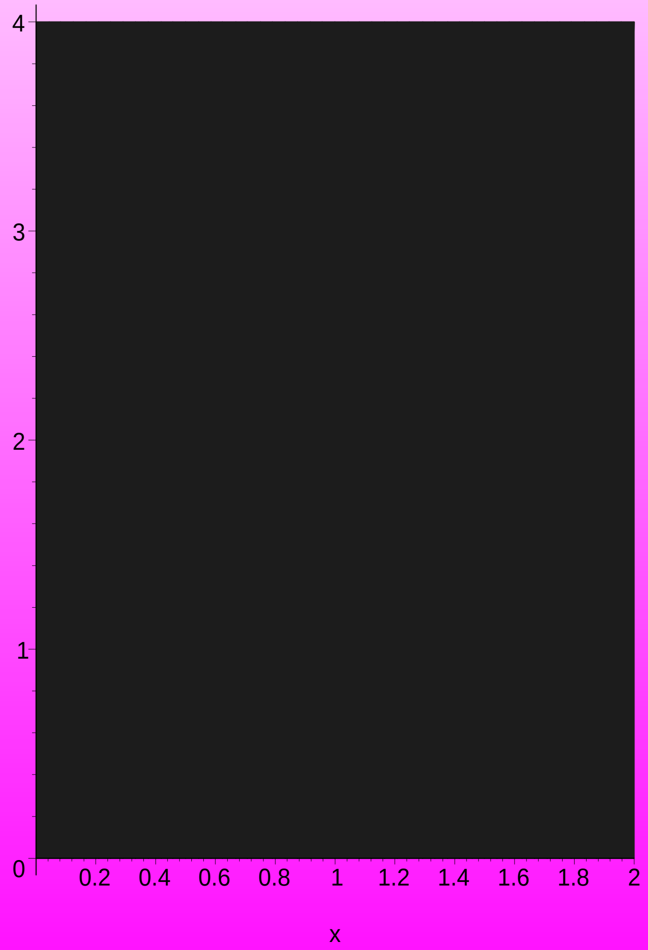
If $f(x) = c = 4$, $a = 0$, and $b = 2$,

$$\int_a^b f(x)dx = c \cdot (b - a) = 4 \cdot (2 - 0) = 8$$

Definite Integral of a Constant Function

If $f(x) = c = 4, \quad a = 0, \quad$ and $b = 2$, the picture would be:



Definite Integral of a Constant Function

Note that if f is constant on $[a, b]$, the definite integral is equal to the Riemann sum using right endpoints with $n = 1$:

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The definite integral is also equal to L_n for any n:

$$\int_a^b f(x)dx = L_n, \quad n = 1, 2, \ldots$$

Definite Integral of a Sum of Functions

Property 2:

If the functions f and g are continuous on an interval $[a, b]$,

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

Definite Integral of a Sum of Functions

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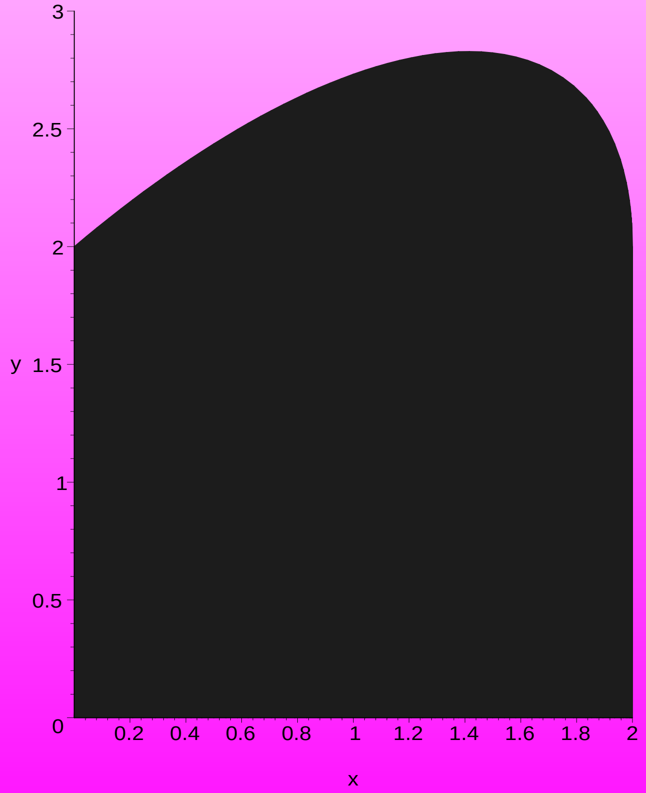
In words, the definite integral of the sum of functions is equal to the sum of their definite integrals.

Definite Integral of a Sum of Functions

Suppose

$$f(x) = x + \sqrt{4 - x^2} \quad 0 \le x \le 2$$

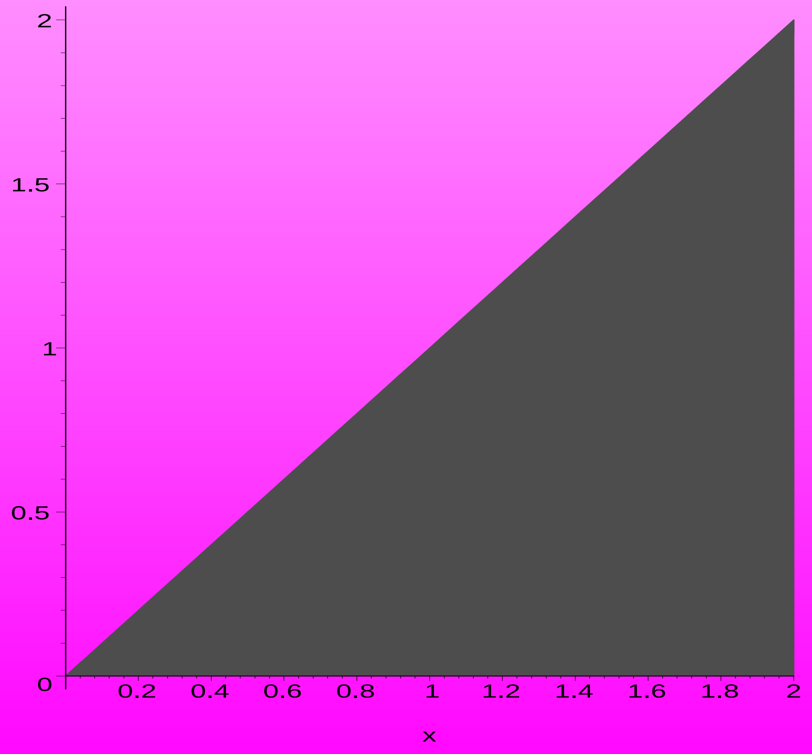
Then the area under the graph from $a = 0$ to $b = 2$ is



Definite Integral of a Sum of Functions

According to the second property, we can write this as a sum of two areas, one for $f(x)$,

$$\int_a^b f(x)dx = \int_0^2 x dx$$

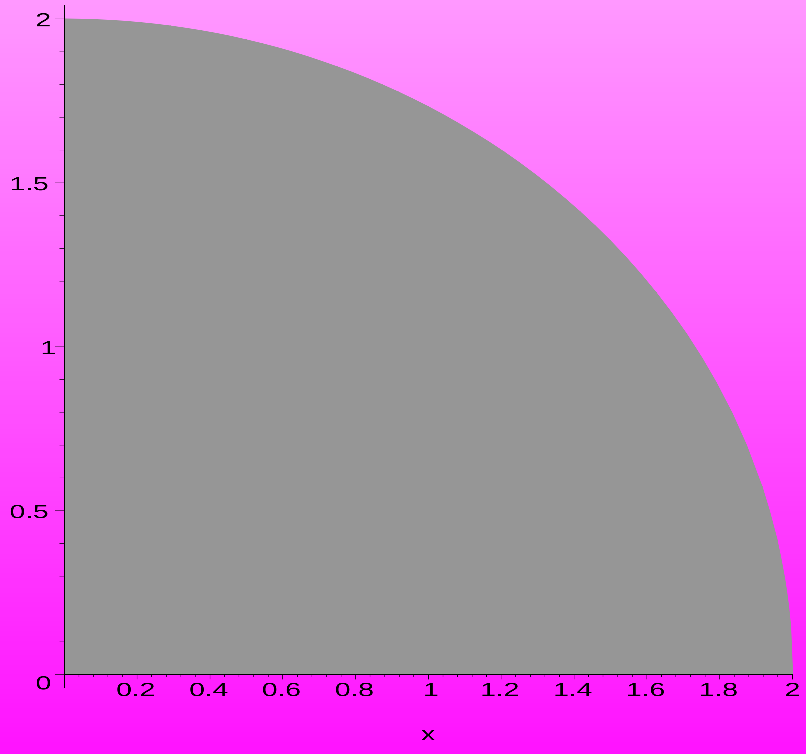


Definite Integral of a Sum of Functions

and one for $g(x)$,

$$\int_a^b g(x)dx = \int_0^2 \sqrt{4 - x^2}dx$$

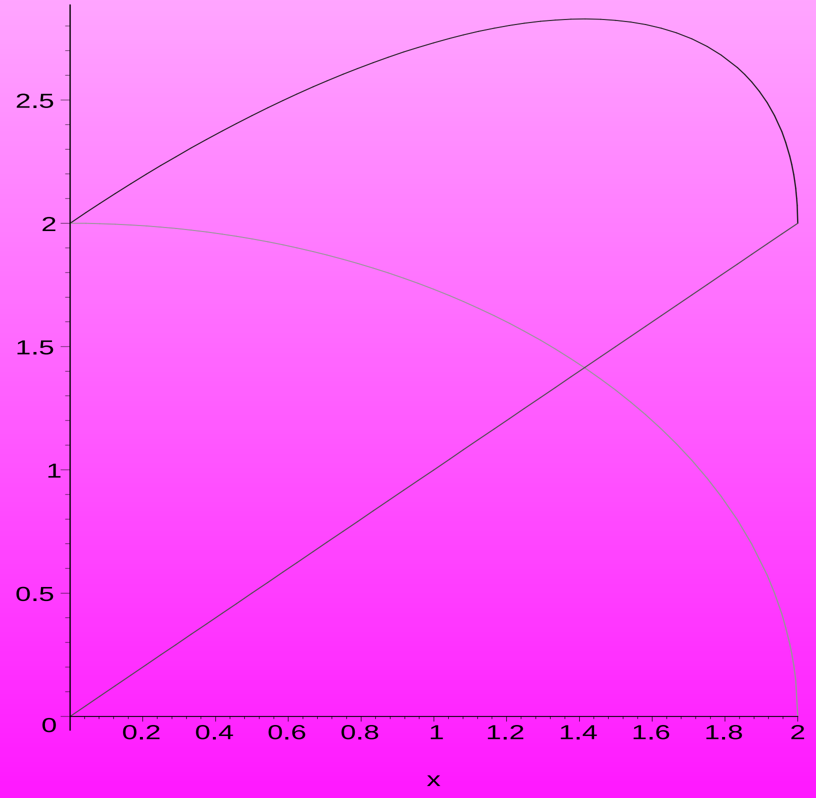
The area under the graph of g from $a = 0$ to $b = 2$ is



Definite Integral of a Sum of Functions

The area graphs reflect the relationship between

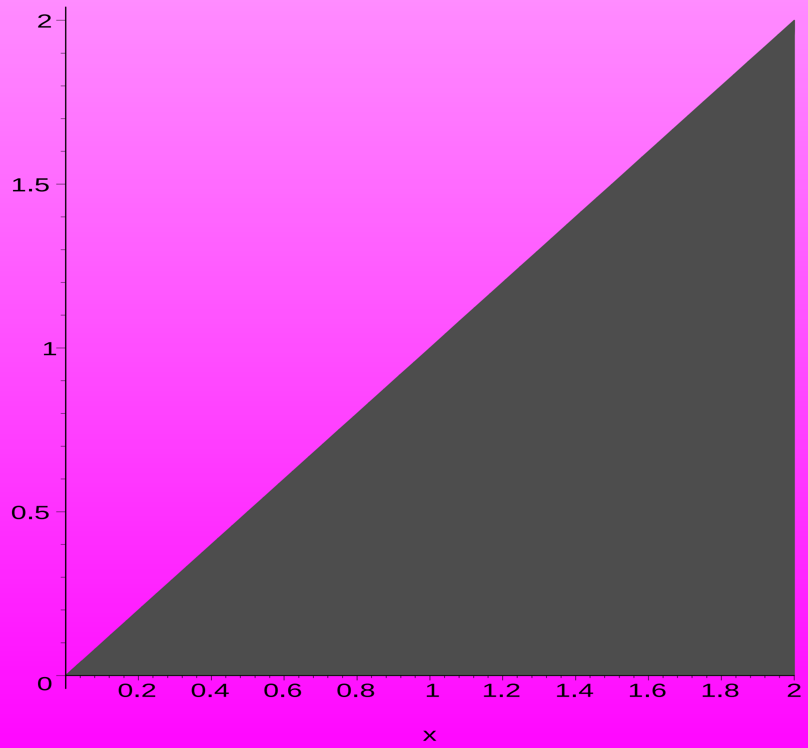
$$f(x), \quad g(x), \quad \text{and} \quad f(x) + g(x)$$



Definite Integral of a Sum of Functions

From elementary geometry, the area under $f(x)$ from a to b is

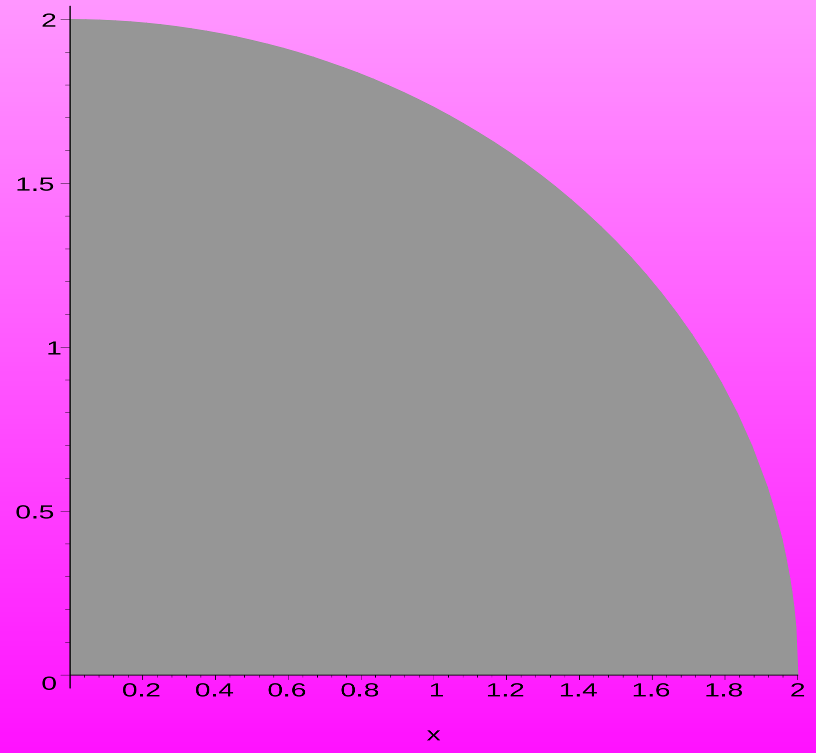
$$\text{Area} = \int_0^2 x\,dx = \frac{1}{2} b \cdot h = \frac{1}{2}(b-a)\cdot f(2) = 2$$



Definite Integral of a Sum of Functions

The area under $g(x)$ is one quarter of a circle of radius 2, so

$$\text{Area} = \int_0^2 \sqrt{4 - x^2}dx = \frac{1}{4}\pi 2^2 = \pi$$



Definite Integral of a Sum of Functions

Property 2 says that

$$\int_0^2 (x + \sqrt{4 - x^2})dx = \int_0^2 xdx + \int_0^2 \sqrt{4 - x^2}dx$$

$$= 2 + \pi$$

Definite Integral of a Constant Times a Function

Property 3:

If the functions f is continuous on an interval $[a, b]$ and c is any constant,

$$\int_a^b c \cdot f(x)dx = c \cdot \int_a^b f(x)dx$$

Definite Integral of a Constant Times a Function

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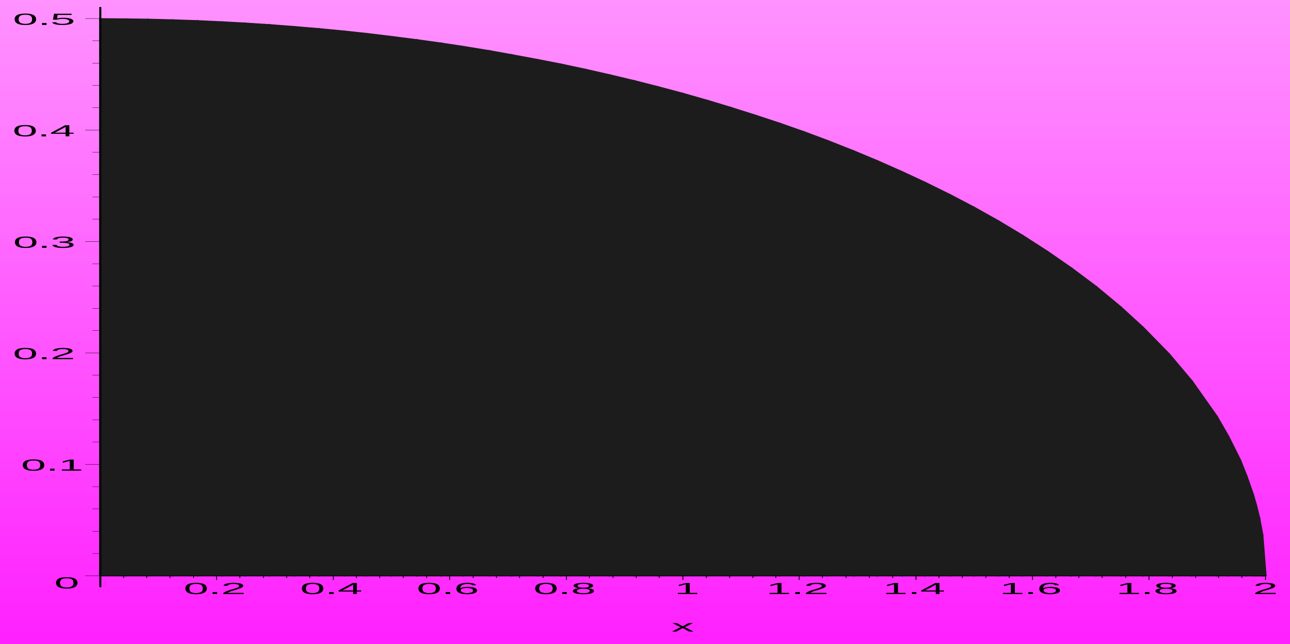
In words, the definite integral of a constant times a function is equal to that constant times the integral of the function.

Definite Integral of a Constant Times a Function

Suppose

$$f(x) = \frac{1}{4}\sqrt{4 - x^2} \quad 0 \leq x \leq 2$$

Then the area under the graph from $a = 0$ to $b = 2$ is one quarter of the area of an ellipse:



Definite Integral of a Sum of Functions

We don't have a formula for the area of an ellipse. However, Property 3 says that the area will be equal to

$$\text{Area} = \int_0^2 \frac{1}{4}\sqrt{4-x^2}dx = \frac{1}{4}\int_0^2 \sqrt{4-x^2}dx$$

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We've already seen that

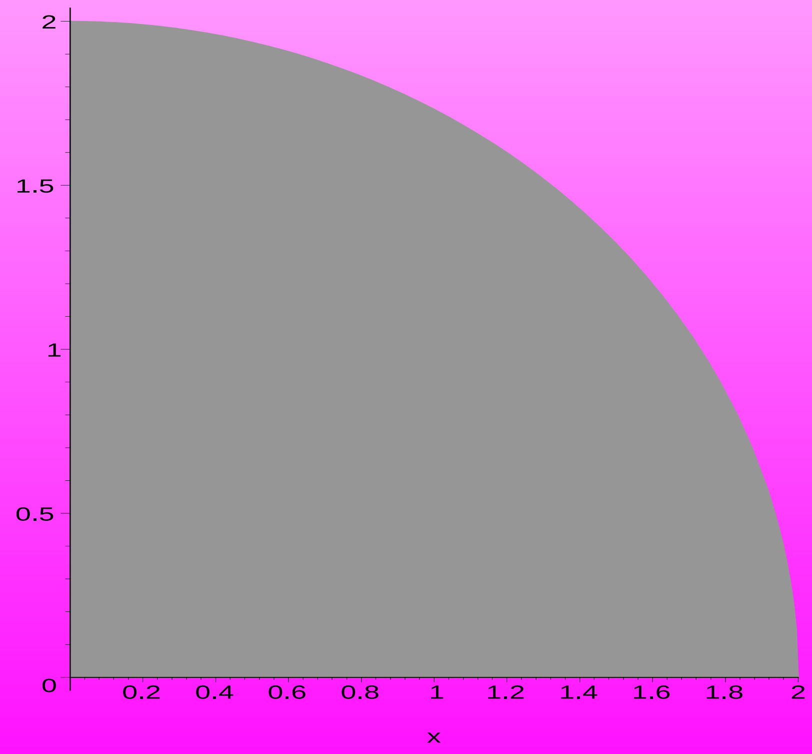
$$\int_0^2 \sqrt{4-x^2}dx$$

represents one quarter of the area of a circle of radius 2.

Definite Integral of a Sum of Functions

So the area under one quarter of the ellipse is

$$\text{Area} = \frac{1}{4}\int_0^2 \sqrt{4 - x^2}dx = \frac{1}{4} \cdot \frac{1}{4}\pi 2^2 = \frac{\pi}{4}$$



Definite Integral of a Difference of Functions

Property 4:

If the functions f and g are continuous on an interval $[a, b]$,

$$\int_a^b (f(x) - g(x))dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

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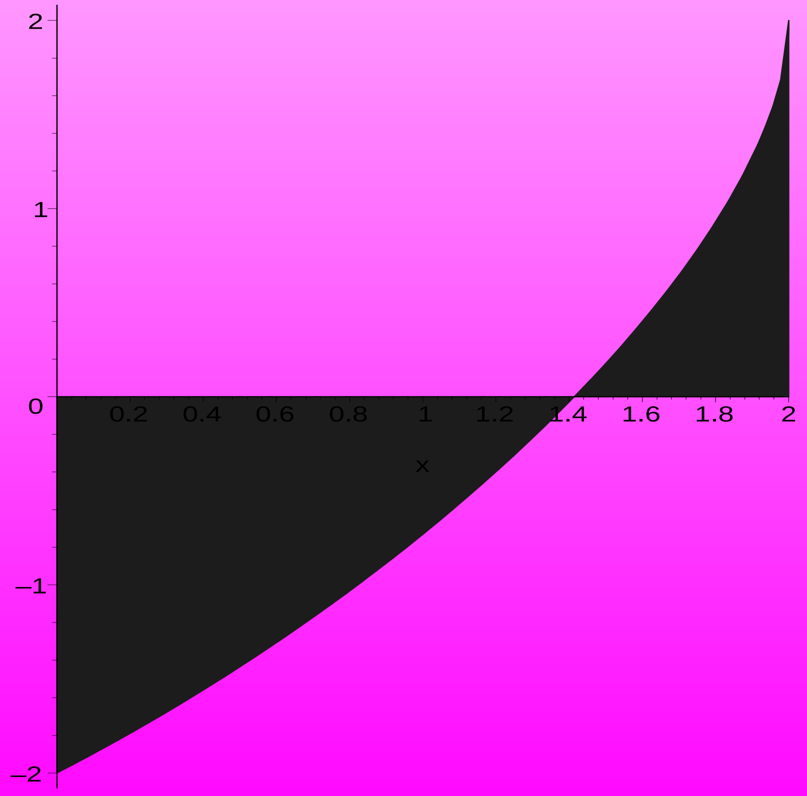
In words, the definite integral of the difference of functions is equal to the difference of their individual definite integrals.

Definite Integral of a Difference of Functions

Suppose we have

$$f(x) = x - \sqrt{4 - x^2} \quad 0 \le x \le 2$$

Then the area under the graph from $a = 0$ to $b = 2$ is



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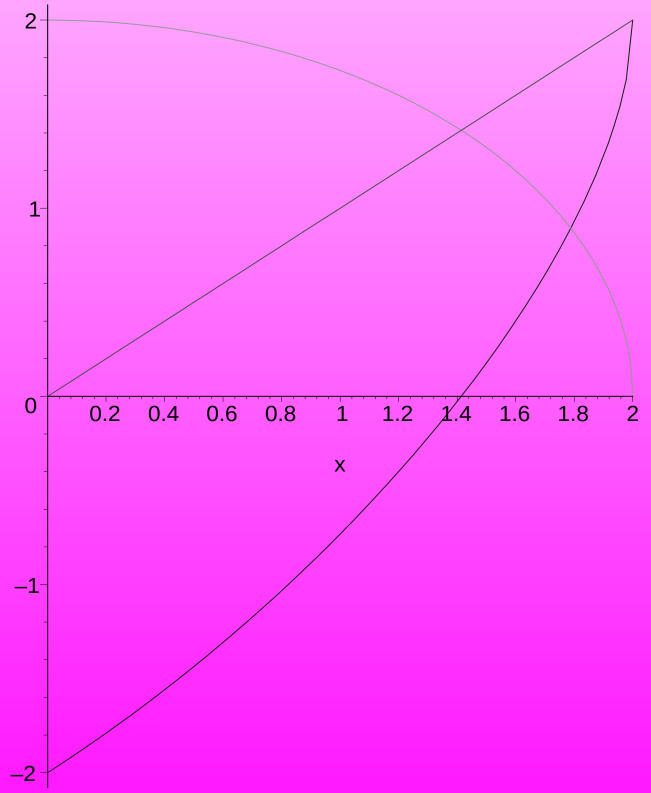

Definite Integral of a Difference of Functions

Property 4 says that

$$\int_0^2 (x - \sqrt{4 - x^2})dx = \int_0^2 x dx - \int_0^2 \sqrt{4 - x^2} dx$$

$$= 2 - \pi$$

Note that the area is **negative**.

Definite Integral of a Difference of Functions

This is a consequence of the fact that the definite integral is the limit of Riemann sums.

Think about what the value of

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

is when $f(x_i^*) < 0$.

Definite Integral Over Adjacent Intervals

The next property says that if we have two definite integrals of the same function over adjacent intervals, we can combine them.

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Property 5:

Suppose f is continuous on the interval $[a, b]$ and c lies between a and b, that is,

$$a \leq c \leq b$$

Then

$$\int_a^c f(x)dx + \int_c^b f(x)dx \quad = \quad \int_a^b f(x)dx$$