The Definite Integral

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Definition of a Definite Integral

Definition: If f is a continuous function on the closed interval $[a, b]$, and we divide that interval into n subintervals

$$\{[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, b]\}$$

each of length

$$\Delta x = \frac{b-a}{n},$$

then choose n *sample points* $x_1^*, x_2^*, \ldots, x_n^*$, one sample point in each interval,

$$x_{i-1} \leq x_i^* \leq x_i, \quad , i = 1, 2, \ldots, n$$

then the **definite integral of** f **from** a **to** b is

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Terminology for Definite Integrals

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The process of calculating an integral is called **integration**.

Notation for a Definite Integral

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That is to say, each of the following represent the same definite integral:

$$\int_a^b f(x)dx \quad \int_a^b f(u)du \quad \int_a^b f(z)dz$$

Evaluating Integrals

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Often this requires one of the following formulas for sums of integers:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Evaluating Integrals

We will start with a simple example. Suppose we want to evaluate

$$\int_0^1 x dx$$

directly from the definition, that is, by finding

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

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In this case,

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

Evaluating Integrals

If we choose x_i^* to be the right endpoint of the i^{th} interval,

$$x_i^* = x_i = i\Delta x = \frac{i}{n}$$

then

$$\int_0^1 x dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{i}{n}\right) \cdot \frac{1}{n}$$

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Since $f(x) = x$, we can substitute to obtain

$$\int_0^1 x dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{i}{n}\right) \cdot \frac{1}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n}$$

Evaluating Integrals

Or:

$$\int_0^1 x dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n^2}$$

$$= \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} i$$

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Using the formula

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

we can write this as

$$\int_0^1 x dx = \lim_{n\to\infty} \frac{1}{n^2} \frac{n(n+1)}{2}$$

Evaluating Integrals

So, evaluating the limit

$$\lim_{n\to\infty} \frac{1}{n^2} \frac{n(n+1)}{2} = \lim_{n\to\infty} \frac{n^2+n}{2n^2} = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{2n}\right)$$

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Since the integral represents the area under the line $y = x$ from $x = 0$ to $x = 1$, we expect the answer to be $1/2$.

Evaluating Integrals

The process becomes more complicated if f is not as simple as it is in this example, but the process remains the same:

- Express the x_i^* in terms of a, b, i, and n.
- Write down the Riemann sum and try to simplify it.
- In particular, try to find function evaluations that result in some multiple of i, i^2, or i^3.
- Replace the sum with one of the summation formulas, and try to find the limit.