4.9a Antiderivatives

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1.Antiderivatives

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Often F is an antiderivative of f for all real numbers x, so

$$I = (-\infty, \infty)$$

In this case F satisfies the definition of an antiderivative for any interval I, and the interval is usually not mentioned.

2.Antiderivatives

Example: Suppose

$$f(x) = 3x^2 \quad \text{and} \quad F(x) = x^3$$

Then F is an antiderivative of f (on any interval you choose) because

$$F'(x) = \frac{d}{dx}x^3 = 3x^2 = f(x)$$

3.Antiderivatives

We say *an* antiderivative rather than *the* antiderivative because $f(x) = 3x^2$ has many antiderivatives on any given interval I:

$$\begin{aligned} F(x) &= x^3 \\ F_2(x) &= x^3 + 10 \\ F_3(x) &= x^3 - 3 \\ F_4(x) &= x^3 + 4 \end{aligned}$$

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Clearly if F is an antiderivative of f on some interval, we can add an arbitrary constant C to F and still have an antiderivative of f.

4.Examples of Antiderivatives

If F is an antiderivative of f on an interval I, then for $x \in I$,

$$c \cdot F(x)$$

is an antiderivative of

$$c \cdot f(x)$$

5.Examples of Antiderivatives

Example: Suppose $f(x) = x^4$ and

$$g(x) \;=\; 4 \cdot f(x)$$

Find an antiderivative of g and its associated interval.

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Find an antiderivative of g and its associated interval.

In this case $F(x) = x^5/5$ is an antiderivative of $f(x)$ on $I = (-\infty, \infty)$, so

$$G(x) \; = \; 4 \cdot F(x) \; = \; \frac{4}{5}x^5$$

is an antiderivative of $g(x)$ on $I = (-\infty, \infty)$.

6.Examples of Antiderivatives

If F is an antiderivative of f and G is an antiderivative of g on an interval I, then on that interval

$$F(x) + G(x)$$

is an antiderivative of

$$f(x) + g(x)$$

7.Examples of Antiderivatives

Example: Suppose

$$f(x) = x \quad \text{and} \quad g(x) = x^2$$

Find an antiderivative of $f + g$ and its associated interval I.

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Note that

- $F(x) = x^2/2$ is an antiderivative of $f(x)$ on $I = (-\infty, \infty)$
- $G(x) = x^3/3$ is an antiderivative of $g(x)$ on $I = (-\infty, \infty)$

so

$$F(x) + G(x) = \frac{x^2}{2} + \frac{x^3}{3}$$

is an antiderivative of $f(x) + g(x)$ on $I = (-\infty, \infty)$.

8.Examples of Antiderivatives

If

$$f(x) = x^n \quad n \neq -1$$

then

$$F(x) = \frac{x^{n+1}}{n+1}$$

is an antiderivative of f.

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We can omit the statement of what the interval I is in this case because the statement is true for any interval I.

9.Examples of Antiderivatives

Example: If

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}$$

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In this case the exponent is $n = 1/2$ so F has the form

$$F(x) \;=\; \frac{x^{\left(\frac{1}{2}+1\right)}}{\left(\frac{1}{2}+1\right)} \;=\; \frac{2}{3} \cdot x^{\frac{3}{2}}$$

The derivative of F is f for all values in $[0, \infty)$ (i.e., the domain of f) so

$$I \;=\; [0, \infty)$$

10.Examples of Antiderivatives

We have seen how to find antiderivatives of functions of the form x^n when $x \neq -1$.

Now consider the case of $n = -1$, or

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Recall from Section 3.7 that

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

This statement is valid on the domain of $\ln x$, which is the interval

$$I = (0, \infty),$$

11.Examples of Antiderivatives

Now we can say that

$$F(x) = \ln x \quad \text{is an antiderivative of} \quad f(x) = \frac{1}{x} \quad \text{on} \quad I = (0, \infty)$$

However, the function $f(x) = 1/x$ is defined everywhere except zero.

Can we find a function F that is an antiderivative of f on its entire domain?

12.Examples of Antiderivatives

Yes, define F as a piecewise function

$$F(x) = \begin{cases} \ln x & \text{if} \quad x > 0 \\ \ln(-x) & \text{if} \quad x < 0 \end{cases}$$

Then $F'(x) = f(x)$ for all $x \neq 0$.

13. Examples of Antiderivatives

Recall the definition of absolute value as a piecewise function:

$$|x| = \begin{cases} x & \text{if} \quad x > 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

This suggests that we may write the antiderivative

$$F(x) = \begin{cases} \ln x & \text{if} \quad x > 0 \\ \ln(-x) & \text{if} \quad x < 0 \end{cases}$$

more succinctly in the equivalent form

$$F(x) = \ln|x|$$