# Series

If we add the first few terms of the infinite sum

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$

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Years ago we defined addition as a binary operation, but our definition says nothing about an infinite sum.

As a result, we really need some new definitions if we want to speak intelligently about infinite sums.

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So we consider the **sequence** $s_1, s_2, s_3, \ldots$ of *partial sums*,

$$\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \\
s_4 &= a_1 + a_2 + a_3 + a_4 \\
\vdots \quad & \quad \vdots
\end{aligned}$$

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$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ s_4 &= a_1 + a_2 + a_3 + a_4 \\ &\vdots \end{aligned}$$

or

$$s_n = \sum_{i=1}^{n} a_i$$

# Series

**Definition**: Given a series $a_1 + a_2 + a_3 + \cdots$, let $s_n$ denote the $n^{th}$ partial sum

$$s_n = \sum_{i=1}^{n} a_i \; = \; a_1 + a_2 + \cdots + a_n$$

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If the sequence $\{s_n\}$ is convergent and there is a real number $s$ such that

$$\lim_{n \to \infty} s_n = s$$

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Otherwise, we say that the series is **divergent**

# Geometric Series

The **geometric series**

$$1 + r + r^2 + r^3 + \cdots = \sum_{i=1}^{\infty} r^{n-1}$$

converges if $|r| < 1$.

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If $|r| \geq 1$, the geometric series is **divergent**

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More generally, the author write the geometric series with a constant multiplier $a$:

$$a1 + ar + ar^2 + ar^3 + \cdots = \sum_{i=1}^{\infty} ar^{n-1} = a \sum_{i=1}^{\infty} r^{n-1}$$

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There are examples of divergent series where $a_n$ converges to zero, and examples where $a_n$ does not converge to zero.

What **can** be said is that if the *sequence* $\{a_n\}$ does **not** converge to zero, then the *series* $a_1 + a_2 + \cdots$ is divergent.

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If $|r| \geq 1$, the geometric series is **divergent**

# Question 1

Determine whether the series converges or diverges. If it converges, find the sum.

$$1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots$$

1. $1$
2. $\frac{4}{5}$
3. $\frac{5}{4}$
4. $\frac{6}{5}$
5. diverges
6. none of the above

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3. $\frac{5}{4}$
4. $\frac{6}{5}$
5. diverges
6. none of the above

3. $\frac{5}{4}$

# Solution

This is a geometric series with

$$r = \frac{1}{5}$$

The series converges because $|r| < 1$. The sum is

$$\sum_{n=1}^{\infty} \frac{1}{5^{(n-1)}} \ = \ \frac{1}{1 - \frac{1}{5}} = \frac{1}{\frac{4}{5}} = \frac{5}{4}$$

# Question 2

Determine whether the series converges or diverges. If it converges, find the sum.

$$1 - \frac{1}{5} + \frac{1}{25} - \frac{1}{125} + \cdots$$

1. $1$
2. $\frac{4}{5}$
3. $\frac{5}{4}$
4. $\frac{5}{6}$
5. diverges
6. none of the above

# Question 2

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1. $1$
2. $\frac{4}{5}$
3. $\frac{5}{4}$
4. $\frac{5}{6}$
5. diverges
6. none of the above

4. $\frac{5}{6}$

# Solution

This is a geometric series with

$$r = -\frac{1}{5}$$

The series converges because $|r| < 1$. The sum is

$$\sum_{n=1}^{\infty} \left(-\frac{1}{5}\right)^{(n-1)} = \frac{1}{1+\frac{1}{5}} = \frac{1}{\frac{6}{5}} = \frac{5}{6}$$

# Question 3

Determine whether the series converges or diverges. If it converges, find the sum.

$$\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$$

| | | | |
|---|---|---|---|
| 1. | $1$ | 4. | $\frac{6}{5}$ |
| 2. | $\frac{4}{5}$ | 5. | diverges |
| 3. | $\frac{5}{4}$ | 6. | none of the above |

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1. $1$
2. $\frac{4}{5}$
3. $\frac{5}{4}$
4. $\frac{6}{5}$
5. diverges
6. none of the above

1. $1$

# Solution

This is a telescoping sum:

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots$$

The series converges because the terms approach zero and only the first and last terms appear in the partial sum,

$$s_n=\frac{1}{1}-\frac{1}{\sqrt{n+1}}$$

so $s_n \to 1$ as $n \to \infty$.

# Question 4

Determine whether the series converges or diverges. If it converges, find the sum.

$$\frac{1}{25} + \frac{1}{125} + \frac{1}{625} + \cdots$$

1. $1$
2. $\frac{1}{20}$
3. $\frac{1}{4}$
4. $\frac{5}{20}$
5. diverges
6. none of the above

# Question 4

Determine whether the series converges or diverges. If it converges, find the sum.

$$\frac{1}{25} + \frac{1}{125} + \frac{1}{625} + \cdots$$

1. $1$
2. $\frac{1}{20}$
3. $\frac{1}{4}$
4. $\frac{5}{20}$
5. diverges
6. none of the above

2. $\frac{1}{20}$

# Solution

This is a geometric series with

$$r = \frac{1}{5} \quad \text{and} \quad a = \frac{1}{25}$$

The series converges because $|r| < 1$. The sum is

$$\sum_{n=1}^{\infty} \frac{1}{25} \frac{1}{5^{(n-1)}} \;=\; \frac{1}{25} \frac{1}{1 - \frac{1}{5}} = \frac{1}{25} \frac{1}{\frac{4}{5}} = \frac{1}{20}$$

# Question 5

Determine whether the series converges or diverges. If it converges, find the sum.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$$

1. $1$
2. $\frac{1}{20}$
3. $\frac{1}{4}$
4. $\frac{5}{20}$
5. diverges
6. none of the above

# Question 5

Determine whether the series converges or diverges. If it converges, find the sum.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$$

1. $1$
2. $\frac{1}{20}$
3. $\frac{1}{4}$
4. $\frac{5}{20}$
5. diverges
6. none of the above

5. diverges

# Solution

This is the harmonic series multiplied by $1/2$, so it diverges.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$$

$$= \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots\right)$$