Sequences

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Another way of thinking of a sequence is a *function* whose domain is the natural numbers $\mathbb{N} = 1, 2, 3, \ldots$.

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Sometimes it is possible to express a sequence in terms of a formula such as:

$$\{a_n\} = \frac{n}{n+2} \quad = \quad \left\{ \frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \dots \right\}$$

or

$$\{x_n\} = \frac{(-1)^n (n+1)}{3^n} \quad = \quad \left\{ \frac{-2}{3}, \frac{3}{9}, \frac{-4}{27}, \frac{5}{81}, \dots \right\}$$

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The formula for the n^{th} term of the preceding sequence is:

$$\{a_n\} = \frac{3n+1}{2n+3}, \quad n = 1, 2, 3, \dots$$

Limits of Sequences

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Definition A sequence $\{a_n\}$ has the **limit** L denoted by

$$\lim_{n \to \infty} a_n = L \quad \text{or} \quad a_n \to L \quad \text{as} \quad n \to \infty$$

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If $\lim a_n$ exists, we say the sequence **converges**. Otherwise, we say it **diverges**.

Precise Definition of a Limit

A more precise definition of the limit of a sequence is the following:

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One way to interpret this definition is the following: No matter how small you choose $\epsilon > 0$, only a finite number of terms of $\{a_n\}$ lie outside the interval

$$(L - \epsilon, L + \epsilon)$$

Sequences

One way to manufacture a sequence from an otherwise continuous function is to consider the domain to consist only of integers:

$$f(n) = a_n \quad \text{where} \quad f(x) = e^x \quad \text{and} \quad n = 1, 2, 3, 4, \ldots$$

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The following theorem shows that the limit as $n \to \infty$ for this sequence is the same as the limit of the function f as $x \to \infty$:

If

$$\lim_{x \to \infty} f(x) = L \quad \text{and} \quad f(n) = a_n, \quad \text{then } \lim a_n = L$$

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If f is continuous and x_n a sequence with $x_n \to L$, then

$$\lim_{n\to\infty} f(x_n) = f(L)$$

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In other words, the processes of taking the limit and evaluating the function can be interchanged:

$$\lim_{n\to\infty} f(x_n) \;=\; f(\lim_{n\to\infty} x_n)$$

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A sequence that is either increasing or decreasing is called **monotonic**

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The sequence

$$\{r^n\} = r, r^2, r^3, r^4, \ldots$$

is convergent if $-1 < r \leq 1$, and divergent otherwise.

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$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if} \quad -1 < r < 1 \\ 1 & \text{if} \quad r = 1 \end{cases}$$

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$\{a_n\}$ is **bounded above** if there is an M such that

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A sequence that is bounded above and below is said to be a **bounded sequence**

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