MA126 Exam 3 (version 1)

Name:

KEY

1) A quantity of protactinium-234 initially has a mass or one gram. If the half life of this isotope is 1.14 minutes, how much remains after 5 minutes have elapsed?

$M(t) = M_0 e^{kt}$

Use the half-life to find k: $.5 = 1.0 e^{k(1.14)} \Rightarrow k = \frac{\ln 0.5}{1.14} = -0.608$

Then:

$M(5) = (1) e^{-.608(5)} = .048\text{ g}$

2) Find the limit of the following sequence, if it exists:

$$\sqrt{5},\ \sqrt{5\sqrt{5}},\ \sqrt{5\sqrt{5\sqrt{5}}}\ldots$$

$\sqrt{5} = 5^{1/2}$ $\quad \sqrt{5\sqrt{5}} = [5 \cdot 5^{1/2}]^{1/2} = (5^{3/2})^{1/2} = 5^{3/4}$

$\sqrt{5\sqrt{5\sqrt{5}}} = [5 \cdot 5^{3/4}]^{1/2} = (5^{7/4})^{1/2} = 5^{7/8}$

$\sqrt{5\sqrt{5\sqrt{5\sqrt{5}}}} = [5 \cdot 5^{7/8}]^{1/2} = (5^{15/8})^{1/2} = 5^{15/16}$

$\therefore \quad a_n = 5^{\left(\frac{2^n-1}{2^n}\right)} = 5^{\left(1 - \frac{1}{2^n}\right)} \rightarrow 5^{(1)} = 5$

The limit is 5.

note: See problem 40 of section 8.1, which was done in class.

3) Find the volume of the solid obtained by rotating about the y-axis the region bounded by the curve $y = \sin x$ and the x-axis between 0 and π.

Using cylindrical shells, the volume is

$$2\pi \int_0^{\pi} x \sin x \, dx$$

using integration by parts, $u = x$, $du = dx$ $dv = \sin x \, dx$ $v = -\cos x$

$$2\pi \int_0^{\pi} x \sin x \, dx = 2\pi\left[-x\cos x \Big|_0^{\pi} - \int_0^{\pi} (-\cos x)\, dx\right] = 2\pi\left[(-x\cos x + \sin x)\Big|_0^{\pi}\right]$$

$$= 2\pi\left[-\pi \cos \pi + 0\cos 0 + \sin \pi - \sin 0\right] = 2\pi(\pi) = 2\pi^2$$

4) Automobiles arrive randomly at a toll booth. If the time between arrivals t (measured in seconds) follows an exponential distribution with probability density function

$$f(t) = \begin{cases} 0 & if \;\; t < 0 \\ \frac{1}{2}\exp(-t/2) & if \;\; t \geq 0 \end{cases}$$

find the average time between arrivals.

The average or mean time between arrivals is:

$$\bar{t} = \frac{\int_{-\infty}^{\infty} t f(t)\, dt}{\int_{-\infty}^{\infty} f(t)\, dt} = \int_{-\infty}^{\infty} t f(t)\, dt$$

In this case,

$$\bar{t} = \int_0^{\infty} t \cdot \frac{1}{2} e^{-t/2}\, dt$$

Let $u = t$, $du = dt$, $dv = \frac{1}{2} e^{-t/2} dt$, $v = -e^{-t/2}$

$$\bar{t} = -te^{-t/2}\Big|_0^{\infty} - \int_0^{\infty} -e^{-t/2}\, dt = \lim_{z \to \infty} -ze^{-z/2} + 2e^{0} - 2e^{-z/2} = 2$$

(See example 3 on page 489 in the text)

5) Use Euler's method with a step size of 0.2 to estimate $y(0.6)$ where $y(x)$ is the solution of the initial-value problem

$$y' = 1 - xy, \qquad y(0) = 0$$

Euler's method: $y_{n+1} = y_n + h \cdot y_n'$, $x_{n+1} = x_n + h$

n	x_n	y_n	y'_n	x_{n+1}	y_{n+1}
0	0	0	1	0.2	0.2
1	0.2	0.2	0.96	0.4	0.392
2	0.4	0.392	0.84	0.6	0.561

$y_1 = y_0 + 0.2(1) = 0 + 0.2 = 0.2$

$y_2 = y_1 + 0.2(.96) = 0.2 + .192$
$= .392$

$y_3 = y(0.6) = y_2 - 0.2(.84) = 0.561$

6) Solve the equation

$$y' = \frac{x\sqrt{x^2+1}}{y \sin y}$$

$\frac{dy}{dx} = \frac{x\sqrt{y^2+1}}{y \sin y} \Rightarrow y \sin y \frac{dy}{dx} = x\sqrt{x^2+1}$

In differential form, $y \sin y \, dy = x\sqrt{x^2+1}\, dx$

Integrate both sides:

$$\int y \sin y \, dy = \int x\sqrt{x^2+1}\, dx$$

$u = y \quad du = dy$
$dV = \sin y \, dy \quad V = -\cos y$

$-y\cos y - \int(-\cos y)\, dy$

$= -y\cos y + \sin y$

$u = x^2+1 \quad \frac{du}{dx} = 2x \quad dx = \frac{du}{2x}$

$= \int \frac{1}{2x} x\sqrt{u}\, du = \frac{1}{2}\int \sqrt{u}\, du = \frac{1}{2}\frac{u^{3/2}}{(3/2)}$

$= \frac{u^{3/2}}{3}$

Solution:

$$\sin y - y\cos y = \frac{(x^2+1)^{3/2}}{3} + C$$

7) Find the length traced by the curve

$$y(t) = \int_1^t e^u \cos u \, du \qquad x(t) = \int_1^t e^u \sin u \, du$$

as t varies from 1 to 2.

$\frac{dy}{dt} = e^t \cos t \quad \frac{dx}{dt} = e^t \sin t$ (By The fundamental Theorem of Calculus)

$$L = \int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} \, dt = \int_1^2 \sqrt{(e^t \cos t)^2 + (e^t \sin t)^2} \, dt$$

$$= \int_1^2 \sqrt{e^{2t}(\cos^2 t + \sin^2 t)} \, dt = \int_1^2 e^t \, dt = e^t \Big|_1^2 = e^2 - e$$

8) For what values of k does the function

$$y = \cos kt$$

satisfy the differential equation

$$4y'' = -25y \ ?$$

$y' = -\sin kt \cdot k \qquad y'' = -\cos kt \cdot k^2$

By substitution,

$4(-\cos kt) k^2 = -25(\cos kt)$

$4k^2(\cos kt) = 25(\cos kt)$

$\Rightarrow 4k^2 = 25$

$k^2 = \frac{25}{4} \qquad k = \pm\sqrt{\frac{25}{4}} = \pm \frac{5}{2}$

9) Determine whether the series

$$\sum_{i=1}^{\infty} \frac{1+(\sqrt{2})^n}{(\sqrt{5})^n}$$

converges or diverges. If it converges, find the sum.

$$\sum_{i=1}^{\infty} \frac{1+(\sqrt{2})^n}{(\sqrt{5})^n} = \sum_{i=1}^{\infty}\left(\frac{1}{\sqrt{5}}\right)^n + \sum_{i=1}^{\infty}\left(\frac{\sqrt{2}}{\sqrt{5}}\right)^n \leftarrow \text{Both geometric series with } r<1$$

$$\sum_{i=1}^{\infty}\left(\frac{1}{\sqrt{5}}\right)^n = \frac{1}{\sqrt{5}} + \left(\frac{1}{\sqrt{5}}\right)^2 + \dots \qquad a = \frac{1}{\sqrt{5}} = r$$

$$\sum_{i=1}^{\infty}\left(\frac{\sqrt{2}}{\sqrt{5}}\right) = \frac{\sqrt{2}}{\sqrt{5}} + \left(\frac{\sqrt{2}}{\sqrt{5}}\right)^2 + \dots \qquad a = \frac{\sqrt{2}}{\sqrt{5}} = r$$

$$\sum_{i=1}^{\infty} \frac{1+(\sqrt{2})^n}{(\sqrt{5})^n} = \left(\frac{\frac{1}{\sqrt{5}}}{1-1/\sqrt{5}}\right) + \frac{\frac{\sqrt{2}}{\sqrt{5}}}{(1-\sqrt{2}/\sqrt{5})}$$

$$= .809 + 4.236 = 5.05$$

10) Find value(s) of b so that the average value of

$$f(x) = x^2 - 9$$

on the interval $[0, b]$ is 3.

$$f_{avg} = \frac{1}{b-0}\int_0^b (x^2-9)\,dx = \frac{1}{b}\left(\frac{x^3}{3} - 9x\right)\Big|_0^b = \frac{b^2}{3} - 9$$

If $f_{avg} = \frac{b^2}{3} - 9 = 3$,

$$b^2 - 27 = 9$$

$$b^2 = 36$$

$$b = \pm 6$$