MA126 Exam 1 (version 1)

Name:

KEY

1) Find

$$\lim_{x\to 0}\frac{1}{x}\int_0^x \cos t\,dt$$

write as: $\lim_{x\to 0} \frac{\int_0^x \cos t\,dt}{x} = \frac{0}{0}$

use l'Hospital's rule: $\lim_{x\to 0} \frac{\int_0^x \cos t\,dt}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}\int_0^x \cos t\,dt}{\frac{d}{dx}x} = \lim_{x\to 0}\frac{\cos x}{1} = 1$

2) An object travels in a straight line with velocity (in ft/sec) given by the piecewise function

$$v(t) = \begin{cases} t^2/3 & \text{if } 0 \le t \le 4 \\ 64 & \text{if } \quad t > 4 \end{cases}$$

How far has the object moved from its initial position (i.e., its position at $t=0$)

a) after 1 second?

b) after 4 seconds?

c) after 10 seconds?

a) $\int_0^1 t^2/3\,dt = t^3/9\Big|_0^1 = 1/9$

b) $\int_0^4 t^2/3\,dt = t^3/9\Big|_0^4 = 64/9$

c) $\int_0^4 t^2/3\,dt + \int_4^{10} 64\,dt = \frac{64}{9} + 6(64)$

3) Evaluate

$$\int_{-\pi}^{\pi}\frac{t}{\sqrt{1+t^6}}\,dt$$

if $f(t) = \frac{t}{\sqrt{1+t^6}}$,

$f(-t) = \frac{(-t)}{\sqrt{1+(-t)^6}} = -\frac{t}{\sqrt{1+t^6}} = -f(t)$

So f is odd and the integral $\int_{-a}^{a} f(x)\,dx = 0$

for any $a > 0$

4) Evaluate the indefinite integral

$$\int \frac{dx}{\sqrt{1-x^2}\,(\sin^{-1} x)}$$

Let $u = \sin^{-1} x$

Then $\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$

So $dx = \sqrt{1-x^2}\,du$ and the integral is $\int \frac{du}{u} = \ln u + C$

$= \ln(\sin^{-1}) + C$

(or $\ln|\sin^{-1}| + C$)

5) Evaluate

$$\int_0^1 t^2 \cdot (1+2t^3)^5\,dt$$

Let $u = 1+2t^3$

Then $\frac{du}{dt} = 6t^2$

and the integral is $\frac{1}{6}\int_{u(0)}^{u(1)} u^5\,du = \frac{u^6}{36}\Big|_1^3$

6) Evaluate

$$\int x^2 \cdot \sin x\,dx$$

use integration by parts (twice)

$u = x^2$

$du = 2x\,dx$

$v = -\cos x$

$dv = \sin x\,dx$

$\int u\,dv = uv - \int v\,du = -x^2\cos x - \int(-\cos x)\,2x\,dx$

$= -x^2\cos x + 2\int x\cos x\,dx$

This time: $u = x$, $du = dx$, $v = \sin x$, $dv = \cos x\,dx$

$2\left(\int x\cos x\,dx\right) = 2\left[x\sin x - \int \sin x\,dx\right]$

$= 2x\sin x + 2\cos x$

The final result is

$\int x^2 \sin x\,dx = -x^2\cos x + 2x\sin x + 2\cos x$

7) Evaluate

$$\int_0^1 \left(\sqrt{1-y^2} + 2y\right)\, dy$$

write as $\int_0^1 \sqrt{1-y^2}\, dy + \int_0^1 2y\, dy$

quadrant of a circle w/ radius = 1

area = $\pi/4$

$\frac{2y^2}{2}\Big|_0^1 = 1$

answer: $1+\pi/4$

8) Use the properties of integrals to show that

$$\frac{\pi}{8} \leq \int_{\pi/4}^{\pi/2} \sin^2 x\, dx \leq \frac{\pi}{4}$$

$\sin\frac{\pi}{4} = 1/\sqrt{2}$ $\quad \sin \pi/2 = 1$

for $\frac{\pi}{4} \leq x \leq \pi/2$, $\frac{1}{\sqrt{2}} \leq \sin x \leq 1$. and, since $\sin x > 0$ in this range, $\frac{1}{2} \leq \sin^2 x \leq 1$

$m \leq f(x) \leq M$

By a comparison property,

$\Rightarrow m(b-a) \leq \int f(x)\,dx \leq M(b-a)$

$\frac{1}{2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) \leq \int_{\pi/4}^{\pi/2} \sin^2 x\, dx \leq 1\left(\frac{\pi}{2}-\frac{\pi}{4}\right)$

$\frac{\pi}{8}$ $\qquad \frac{\pi}{4}$

9) Suppose

$$f(x) = \int_0^x g(t)\,dt \quad \text{and} \quad g(t) = \int_0^{2t} \sin y\,dy$$

Find the second derivative of f with respect to x, d^2f/dx^2.

$$\frac{d}{dx} f(x) = g(x) = \int_0^{2t} \sin y\, dy$$

$$\frac{d^2 f(x)}{dx^2} = \frac{d\,g(x)}{dx} = \sin 2x\,(2) = 2\sin 2x$$

10) Evaluate the definite integral

$$\int_0^1 \frac{1+y}{e^{2y}}\,dy$$

write as: $\int_0^1 e^{-2y}\,dy + \int_0^1 y e^{-2y}\,dy$

↑ Substitution Rule ↑ integration by parts

$$-\frac{1}{2} e^{-2y}\Big|_0^1 + -\frac{1}{4}(1+2y)e^{-2y}\Big|_0^1$$

$$\frac{1}{2} - \frac{1}{2}e^{-2} + \frac{1}{4} - \frac{3}{4}e^{-2}$$

$$= \frac{3}{4} - \frac{5}{4}e^{-2}$$