Review of Trigonometric Functions

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Trigonometric Functions - A Brief Ove

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$$x = \cos\theta \quad \text{and} \quad y = \sin\theta$$

From this definition, the single most important trigonometric identity follows by the Pythagorean theorem: For any θ,

$$\cos^2\theta + \sin^2\theta = 1$$

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In particular, the $\sin^2\theta$ notation should not be confused with

$$\sin\theta^2 \quad \text{and} \quad \sin(\sin\theta)$$

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If we start with the identity

$$\cos^2\theta \ + \ \sin^2\theta \ = \ 1$$

and, assuming $\cos\theta \neq 0$, divide both sides by $\cos^2\theta$, we obtain a second identity,

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Using the definitions of $\tan\theta$ and $\sec\theta$ we can write this as:

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Using the definitions of $\cot\theta$ and $\csc\theta$ we can write this as:

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Trigonometric Sum and Difference For

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The formulas for the $\sin$ and $\cos$ functions of the sum and difference of two angles can be easily derived from an important identity known as **Euler's Formula**.

Euler's Formula states that, for any real number y,

$$e^{iy} = \cos y + i \cdot \sin y$$

where i is imaginary unit, that is, the complex number with the property that

$$i^2 = -1$$

Trigonometric Sum and Difference For

Now suppose θ is replaced by the sum of two angles, θ_1 and θ_2.

Euler's Formula becomes:

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But adding exponents is equivalent to multiplying, so the exponential can be written as:

$$e^{i(\theta_1+\theta_2)} \;=\; e^{i\theta_1}\cdot e^{i\theta_2}$$

Trigonometric Sum and Difference For

Applying Euler's Formula separately to each exponential, we obtain

$$e^{i\theta_1} \cdot e^{i\theta_2} \; = \; (\cos\theta_1 + i \cdot \sin\theta_1) \cdot (\cos\theta_2 + i \cdot \sin\theta_2)$$

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Expanding the binomial products on the right hand side gives:

$$(\cos\theta_1 + i \cdot \sin\theta_1) \cdot (\cos\theta_2 + i \cdot \sin\theta_2) \;=$$

$$=\; (\cos\theta_1 \cos\theta_2 + i^2 \sin\theta_1 \sin\theta_2) + i(\cos\theta_1 \sin\theta_2 + \cos\theta_2 \sin\theta_1)$$

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Since $i^2 = -1$, this becomes

$$=\; (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\cos\theta_1 \sin\theta_2 + \cos\theta_2 \sin\theta_1)$$

Trigonometric Sum and Difference For

Now we have two equivalent expressions:

$$e^{i(\theta_1+\theta_2)} \;=\; \cos(\theta_1+\theta_2) \;+\; i\cdot\sin(\theta_1+\theta_2)$$

$$e^{i\theta_1}\cdot e^{i\theta_2} \;=\; (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)$$

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In each right hand side, the real part is $\cos(\theta_1+\theta_2)$, so equating the real parts of the two expressions, we have:

$$\cos(\theta_1+\theta_2) \; = \; \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$$

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In each right hand side, the real part is $\cos(\theta_1+\theta_2)$, so equating the real parts of the two expressions, we have:

$$\cos(\theta_1+\theta_2) \; = \; \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$$

The imaginary part is $i \cdot \sin(\theta_1+\theta_2)$. Equating the imaginary parts these expressions gives

$$\sin(\theta_1+\theta_2) \; = \; \cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2$$

Trigonometric Sum and Difference For

In summary, using Euler's Formula we have derived the following expressions for the cosine and sine of a sum of two angles:

$$\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$$

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We obtain the double angle formulas in the special case $\theta_1 = \theta_2$:

$$\cos(2\theta) \;=\; \cos^2\theta \;-\; \sin^2\theta$$

$$\sin(2\theta) \;=\; 2\cos\theta\sin\theta$$

Trigonometric Difference Formulas

It remains to find the formula for the sine of the difference of two angles:

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$$\sin(\theta_1 - \theta_2) \;=\; \sin(\theta_1 + (-\theta_2)),$$

by substitution we get:

$$\sin(\theta_1 + (-\theta_2)) \;=\; \sin\theta_1 \cos(-\theta_2) + \cos\theta_1 \sin(-\theta_2)$$

or

$$\sin(\theta_1 - \theta_2) \;=\; \sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2$$

Tangent Sums and Differences

We can use the sum formulas for sine and cosine to derive the formulas for the tangent.

$$\tan(\theta_1 + \theta_2) \;=\; \frac{\sin(\theta_1 + \theta_2)}{\cos(\theta_1 + \theta_2)}$$

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Dividing the numerator and denominator by $\cos\theta_1 \cos\theta_2$ gives:

$$\tan(\theta_1 + \theta_2) = \frac{\frac{\sin\theta_1 \cos\theta_2}{\cos\theta_1 \cos\theta_2} + \frac{\cos\theta_1 \sin\theta_2}{\cos\theta_1 \cos\theta_2}}{\frac{\cos\theta_1 \cos\theta_2}{\cos\theta_1 \cos\theta_2} - \frac{\sin\theta_1 \sin\theta_2}{\cos\theta_1 \cos\theta_2}}$$

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The expressions simplify to:

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Using the definition of $\tan\theta$, this becomes

$$\tan(\theta_1+\theta_2) = \frac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\tan\theta_2}$$

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Using the definition of $\tan\theta$, this becomes

$$\tan(\theta_1+\theta_2) = \frac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\tan\theta_2}$$

A similar argument will show that

$$\tan(\theta_1-\theta_2) = \frac{\tan\theta_1-\tan\theta_2}{1+\tan\theta_1\tan\theta_2}$$