Example 1

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If the two bounding functions converge to the same limit, then the middle function must also converge to that limit.

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However, we know that

$$-1 \leq \sin \theta \leq 1 \quad \text{for all } \theta \in \mathbb{R}$$

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Consequently we can write the double inequality

$$\sqrt{x} \cdot (-1) \leq \sqrt{x}\sin\frac{1}{x} \leq \sqrt{x} \cdot (+1)$$

or

$$-\sqrt{x} \leq \sqrt{x}\sin\frac{1}{x} \leq \sqrt{x}$$

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We know from the limit rules that

$$\lim_{x\to 0} -\sqrt{x} \;=\; \lim_{x\to 0} \sqrt{x}$$

so the leftmost and rightmost terms of the double inequality

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tend to zero as $x \to 0$.

The squeeze theorem states that the middle term must have the same limit, so:

$$\lim_{x \to 0} \sqrt{x} \sin \frac{1}{x} = 0$$

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Again use the squeeze theorem, together with the properties of the $\sin$ function:

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$$(-1) \cdot \tan x \le \tan x \sin \frac{1}{x^2} \le (+1) \cdot \tan x$$

As $x \to 0$, $\tan x \to 0$ so by the squeeze theorem

$$0 \le \lim_{x \to 0} \tan x \sin \frac{1}{x^2} \le 0$$